Extending a Putnam Problem

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Putnam Exam

- The William Lowell Putnam Mathematical Competition is the leading mathematics competition for undergraduate students in the United States and Canada.
- The Putnam exams began in 1938 as a competition between mathematics departments at different colleges.
- Set up:
  - Two 3-hour sessions
  - A1-A6
  - B1-B6

B1 from 1973

- Focusing on problem B1 from the 1973 Putnam Exam.
- B1 is usually easiest problem
- This year A1 and B1 were unusually difficult

B1. S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal.

Introduction

- Among the Putnam exam problems that generated significant interest in the subsequent years is the problem B1 from 1973 ("*2n+1* problem" from here on):
- *"S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal."*
- It has been noticed and actually proved that the *2n+1* problem is still valid if the word "integers" is replaced with "real numbers".
- In the present article will prove a generalization of the above stated Putnam problem. Our main result splits into m groups of n (compared to 2 groups of n as in the *2n+1* problem) once any arbitrary set of r elements is removed (compared to one element as in the *2n+1* problem) and is valid over any field L of characteristic zero. The proof will be provided in Section 2.

Generalization

Theorem 1: *Let* $m \geq 2,\ n, r \geq 1$, *and* $N = mn + r$. *If* $x_1, x_2, \ldots, x_N$ *are elements of a field* L *of characteristic zero with the property that no matter which* r *of the* x_i*'s are removed the remaining* mn *elements can be split into* m *groups of size* n *with equal sums, then* $x_1 = x_2 = \ldots = x_N$.

Counterexample in finite characteristic

Interestingly, we will see that a similar statement to that in Theorem 1 generally fails in fields of prime characteristic. As a quick counterexample, if $m = n = 2$ and $r = 1$, it can be seen that the elements $0, 1, 2, 3, 4 \in F_5$ have the property that no matter which one is removed, the remaining four can be split into two groups of two with equal sums in the finite field F_5. More explicitly, the possible splits are $\{0\} \cup \{1,4\} \cup \{2,3\}$, $\{1\} \cup \{0,2\} \cup \{3,4\}$, $\{2\} \cup \{1,3\} \cup \{0,4\}$, $\{3\} \cup \{2,4\} \cup \{0,1\}$ and $\{4\} \cup \{0,3\} \cup \{1,2\}$.

Proof of Theorem 1

In this section we will prove the validity of Theorem 1 (involving partitions into m groups of n when any r of the x_i's are removed) over any field L of characteristic 0, in other words any field that contains $\mathbb{Q}$ as a subfield.
This will be done gradually over the ascending set of domains $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset L$.

CASE 1: $x_1, x_2, \ldots, x_N$ are nonnegative integers. Then the proof will proceed by induction over the maximum value $M = \max\left(x_1, x_2, \ldots, x_N\right)$. Let $S = x_1 + x_2 + \ldots + x_N$.

But first, let us begin by noting that since the removal of any r of the x_i's, say $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$, allows for the split of the remaining mn elements x_j into m groups of n with equal sums, it follows that

$$(1) \quad S - \left(x_{i_1} + x_{i_2} + \ldots + x_{i_r}\right) \equiv 0 \pmod{m}$$

for all choices of r distinct indices $i_1, i_2, \ldots, i_r$. From (1) we find that the sums $x_{i_1} + x_{i_2} + \ldots + x_{i_r} \bmod m$ of any r elements x_i out of $x_1, x_2, \ldots, x_N$ are the same. The following Lemma would be essential in the unfolding of the induction proof.

Lemma: $x_1 \equiv x_2 \equiv \ldots \equiv x_N \pmod{m}$.

Proof of lemma- see paper

Concluding Case 1

Now that the Lemma is proved, the proof of Case 1 will be done by induction over the maximum value $M = \max(x_1, x_2, \ldots, x_N)$.

If $M = 0$ there is nothing to prove, since then $x_1 = x_2 = \ldots = x_N = 0$. Let $M > 0$ and let $\rho := x_i \bmod m$ - the same for all i ($0 \le \rho \le m-1$). Then it is easy to see that the elements $x_1' = \frac{x_1 - \rho}{m}, x_2' = \frac{x_2 - \rho}{m}, \ldots, x_N' = \frac{x_N - \rho}{m}$ have the same property as $x_1, x_2, \ldots, x_N$, in that the removal of any r of the x_i's allows for the split of the remaining mn elements x_j' into m groups of n with equal sums. Here the assumption that the groups are equal in size is essential. Since $\max(x_1', x_2', \ldots, x_N') < M$, the inductive hypothesis yields $x_1' = x_2' = \ldots = x_N'$, and hence $x_1 = x_2 = \ldots = x_N$.

CASE 2: $x_1, x_2, ..., x_N \in \mathbb{Z}$. This will be proved by a reduction to Case 1. To that effect, let A be a positive integer such that $x_1' = x_1 + A, x_2' = x_2 + A, ..., x_N' = x_N + A$ are natural numbers. Clearly, $x_1', x_2', ..., x_N'$ have the same property as $x_1, x_2, ..., x_N$, in that the removal of any r of the x_i's allows for the split of the remaining mn elements x_j' into m groups of n with equal sums. From Case 1, $x_1' = x_2' = ... = x_N'$, and hence $x_1 = x_2 = ... = x_N$.

CASE 3: $x_1, x_2, ..., x_N \in \mathbb{Q}$. This will be proved by a reduction to Case 2. To that effect, let B be an integer such that $x_1' = Bx_1, x_2' = Bx_2, ..., x_N' = Bx_N$ are integers having the property that the removal of any r of the x_i's allows for the split of the remaining mn elements x_j' into m groups of n with equal sums. From Case 2, $x_1' = x_2' = ... = x_N'$, and hence $x_1 = x_2 = ... = x_N$.

CASE 4: $x_1, x_2, ..., x_N \in L$. This will be proved by a reduction to Case 3 by using the coordinates of $x_1, x_2, ..., x_N$ with respect to a basis of $L/\mathbb{Q}$.

Details- see paper

Finite Characteristic Case

We see that in finite characteristic, the "m groups of n" type of statements such as the one proved in Theorem 1 for fields of characteristic 0, is not necessarily true. The following theorem provides a class of counterexamples when $mn + 1$ is a prime p. Note that according to Dirichlet's Theorem for prime numbers in arithmetic progressions (see [2, Chapter 7]), for every $m \geq 2$, there are infinitely many primes p such that $p \equiv 1 \pmod{m}$, that is primes of the form $p = mn + 1$.

Generalization

Theorem 2. *Let $m \geq 2$, $n \geq 1$ be integers and assume that $p := mn + 1$ is a prime. Let g be a primitive root modulo p. Consider the partition of the finite prime field $F_p = \{0, 1, \ldots, p-1\}$ consisting of $\{0\}$ and the m size n cosets in F_p^* of the multiplicative group of the nonzero mth powers $\{g^m, g^{2m}, \ldots, g^{nm} = 1\}$:*

$$\{0\} \cup \bigcup_{s=0}^{m-1} \{g^{s+m}, g^{s+2m}, \ldots, g^{s+nm}\}. \qquad (5)$$

Then the modular translations in F_p of the partition (5) cover all instances in which an element of the set F_p is removed, while the remaining mn elements are split into m groups of n with equal sums. Thus they provide a counterexample in characteristic p of the statement proved in Theorem 1 for characteristic 0.

Proof Details- see paper

Example

Example. This is a practical, hands-on way to construct the counterexample for small values of the parameters. Let $m = 3$, $n = 4$. Then $p = mn + 1 = 13$. As a primitive root modulo 13, we take $g = 2$.

Let us list the elements of F_{13}^* as powers of the primitive root 2:

K	1	2	3	4	5	6	7	8	9	10	11	12
$2^k \bmod 13$	2	4	8	3	6	12	11	9	5	10	7	1

To get our 3 blocks of 4 upon the removal of 0, we take every third element in the row of the powers (second row) of the above table, so the three groups are

$$\{2, 3, 11, 10\}, \{4, 6, 9, 7\}, \text{ and } \{8, 12, 5, 1\}. \tag{7}$$

Example cont.

Thus if we remove the element 0 out of F_{13}, then the remaining 12 elements can be split into the above 3 groups of 4 with equal sums (the equal sums are zero in this case). What about removing other elements? Say, for example, we remove $2 \in F_{13}$. As indicated in the proof of Theorem 2, we add 2 to each element of a block in (7). Therefore, if we remove the element 2 out of F_{13}, then the remaining 12 elements can be split into the following 3 groups of size 4 with equal sums:

$$\{4, 5, 0, 12\}, \{6, 8, 11, 9\}, \text{ and } \{10, 1, 7, 3\}.$$

As it is easy to see, the sum in each group is 8, which is consistent with the previous construction given in Theorem 2 $(na = 4 \cdot 2 = 8)$. Table 1 below displays an interesting combinatorial pattern, showing all partitions in 3 groups of size 4 occurring when each element of F_{13} is removed.

Element removed	Group 1	Group 2	Group 3	Shift	Sum per group
0	2, 3, 10, 11	4, 6, 7, 9	1, 5, 8, 12	0	0
1	3, 4, 11, 12	5,7, 8, 10	0, 2, 6, 9	1	4
2	0, 4, 5, 12	6, 8, 9, 11	1, 3, 7, 10	2	8
3	0, 1, 5, 6	7, 9, 10, 12	2, 4, 8, 11	3	12
4	1, 2, 6, 7	0, 8, 10, 11	3, 5, 9, 12	4	3
5	2, 3, 7, 8	1, 9, 11, 12	0, 4, 6, 10	5	7
6	3, 4, 8, 9	0, 2, 10, 12	1, 5, 7, 11	6	11
7	4, 5, 9, 10	0, 1, 3, 11	2, 6, 8, 12	7	2
8	5, 6, 10, 11	1, 2, 4, 12	0, 3, 7, 9	8	6
9	6, 7, 11, 12	0, 2, 3, 5	1, 4, 8, 10	9	10
10	0, 7, 8, 12	1, 3, 4, 6	2, 5, 9, 11	10	1
11	0, 1, 8, 9	2, 4, 5, 7	3, 6, 10, 12	11	5
12	1, 2, 9, 10	3, 5, 6, 8	0, 4, 7, 11	12	9

Table 1 shows that if any element of F_{13} is removed, then the remaining 12 can be split into 3 groups of 4 with equal sums.

Main reference:

Mihai Caragiu and Rachael Harbaugh

Extending a Putnam Problem To Fields of Various Characteristics

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