Extending a Putnam Problem

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The William Lowell Putnam Mathematical Competition is the leading mathematics competition for undergraduate students in the United States and Canada.

The Putnam exams began in 1938 as a competition between mathematics departments at different colleges.

Set up:
- Two 3-hour sessions
- A1-A6
- B1-B6
B1 from 1973

- Focusing on problem B1 from the 1973 Putnam Exam.
- B1 is usually easiest problem
- This year A1 and B1 were unusually difficult

**B1.** S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal.
Among the Putnam exam problems that generated significant interest in the subsequent years is the problem B1 from 1973 ("2n+1 problem" from here on):

- "S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal."

- It has been noticed and actually proved that the 2n+1 problem is still valid if the word “integers” is replaced with “real numbers”.

- In the present article will prove a generalization of the above stated Putnam problem. Our main result splits into m groups of n (compared to 2 groups of n as in the 2n+1 problem) once any arbitrary set of r elements is removed (compared to one element as in the 2n+1 problem) and is valid over any field L of characteristic zero. The proof will be provided in Section 2.
Theorem 1: Let $m \geq 2$, $n, r \geq 1$, and $N = mn + r$. If $x_1, x_2, ..., x_N$ are elements of a field $F$ of characteristic zero with the property that no matter which $r$ of the $x_i$’s are removed the remaining $mn$ elements can be split into $m$ groups of size $n$ with equal sums, then $x_1 = x_2 = ... = x_N$. 
Counterexample in finite characteristic

Interestingly, we will see that a similar statement to that in Theorem 1 generally fails in fields of prime characteristic. As a quick counterexample, if \( m = n = 2 \) and \( r = 1 \), it can be seen that the elements \( 0, 1, 2, 3, 4 \in F_5 \) have the property that no matter which one is removed, the remaining four can be split into two groups of two with equal sums in the finite field \( F_5 \). More explicitly, the possible splits are:

\[
\{0\} \cup \{1, 4\} \cup \{2, 3\}, \{1\} \cup \{0, 2\} \cup \{3, 4\}, \{2\} \cup \{1, 3\} \cup \{0, 4\}, \{3\} \cup \{2, 4\} \cup \{0, 1\} \text{ and } \{4\} \cup \{0, 3\} \cup \{1, 2\}.
\]
Proof of Theorem 1

In this section we will prove the validity of Theorem 1 (involving partitions into \( m \) groups of \( n \) when any \( r \) of the \( x_i \)'s are removed) over any field \( L \) of characteristic 0, in other words any field that contains \( \mathbb{Q} \) as a subfield.
This will be done gradually over the ascending set of domains \( \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset L \).

CASE 1: \( x_1, x_2, ..., x_N \) are nonnegative integers. Then the proof will proceed by induction over the maximum value \( M = \max \left( x_1, x_2, ..., x_N \right) \). Let \( S = x_1 + x_2 + ... + x_N \).

But first, let us begin by noting that since the removal of any \( r \) of the \( x_i \)'s, say \( x_{i_1}, x_{i_2}, ..., x_{i_r} \), allows for the split of the remaining \( mn \) elements \( x_j \) into \( m \) groups of \( n \) with equal sums, it follows that

\[
S - \left( x_{i_1} + x_{i_2} + ... + x_{i_r} \right) \equiv 0 \pmod{m}
\]

for all choices of \( r \) distinct indices \( i_1, i_2, ..., i_r \). From (1) we find that the sums \( x_{i_1} + x_{i_2} + ... + x_{i_r} \mod m \) of any \( r \) elements \( x_j \) out of \( x_1, x_2, ..., x_N \) are the same. The following Lemma would be essential in the unfolding of the induction proof.

Lemma: \( x_1 \equiv x_2 \equiv ... \equiv x_N \pmod{m} \). Proof of lemma- see paper
Concluding Case 1

Now that the Lemma is proved, the proof of Case 1 will be done by induction over the maximum value $M = \max\left(x_1, x_2, \ldots, x_N\right)$.

If $M = 0$ there is nothing to prove, since then $x_1 = x_2 = \ldots = x_N = 0$. Let $M > 0$ and let $\rho := x_i \mod m$ - the same for all $i \, (0 \leq \rho \leq m - 1)$. Then it is easy to see that the elements

$$x'_1 = \frac{x_1 - \rho}{m}, \quad x'_2 = \frac{x_2 - \rho}{m}, \quad \ldots, \quad x'_N = \frac{x_N - \rho}{m}$$

have the same property as $x_1, x_2, \ldots, x_N$, in that the removal of any $r$ of the $x'_i$'s allows for the split of the remaining $mn$ elements $x'_j$ into $m$ groups of $n$ with equal sums. Here the assumption that the groups are equal in size is essential. Since $\max\left(x'_1, x'_2, \ldots, x'_N\right) < M$, the inductive hypothesis yields $x'_1 = x'_2 = \ldots = x'_N$, and hence $x_1 = x_2 = \ldots = x_N$. 
CASE 2: $x_1, x_2, ..., x_N \in \mathbb{Z}$. This will be proved by a reduction to Case 1. To that effect, let $A$ be a positive integer such that $x_1' = x_1 + A, x_2' = x_2 + A, ..., x_N' = x_N + A$ are natural numbers. Clearly, $x_1', x_2', ..., x_N'$ have the same property as $x_1, x_2, ..., x_N$, in that the removal of any $r$ of the $x_i$'s allows for the split of the remaining $mn$ elements $x_j'$ into $m$ groups of $n$ with equal sums. From Case 1, $x_1' = x_2' = ... = x_N'$, and hence $x_1 = x_2 = ... = x_N$.

CASE 3: $x_1, x_2, ..., x_N \in \mathbb{Q}$. This will be proved by a reduction to Case 2. To that effect, let $B$ be an integer such that $x_1' = Bx_1, x_2' = Bx_2, ..., x_N' = Bx_N$ are integers having the property that the removal of any $r$ of the $x_i$'s allows for the split of the remaining $mn$ elements $x_j'$ into $m$ groups of $n$ with equal sums. From Case 2, $x_1' = x_2' = ... = x_N'$, and hence $x_1 = x_2 = ... = x_N$.

CASE 4: $x_1, x_2, ..., x_N \in L$. This will be proved by a reduction to Case 3 by using the coordinates of $x_1, x_2, ..., x_N$ with respect to a basis of $L/\mathbb{Q}$. Details- see paper.
Finite Characteristic Case

We see that in finite characteristic, the “$m$ groups of $n$” type of statements such as the one proved in Theorem 1 for fields of characteristic 0, is not necessarily true. The following theorem provides a class of counterexamples when $mn + 1$ is a prime $p$. Note that according to Dirichlet’s Theorem for prime numbers in arithmetic progressions (see [2, Chapter 7]), for every $m \geq 2$, there are infinitely many primes $p$ such that $p \equiv 1 \pmod{m}$, that is primes of the form $p = mn + 1$. 
Generalization

Theorem 2. Let $m \geq 2, \ n \geq 1$ be integers and assume that $p := mn + 1$ is a prime. Let $g$ be a primitive root modulo $p$. Consider the partition of the finite prime field $F_p = \{0, 1, \ldots, p - 1\}$ consisting of $\{0\}$ and the $m$ size $n$ cosets in $F_p^*$ of the multiplicative group of the nonzero $m$th powers $\{g^m, g^{2m}, \ldots, g^{nm} = 1\}$:

$$\{0\} \cup \bigcup_{s=0}^{m-1} \{g^{s+m}, g^{s+2m}, \ldots, g^{s+nm}\}.$$  \hspace{1cm} (5)

Then the modular translations in $F_p$ of the partition (5) cover all instances in which an element of the set $F_p$ is removed, while the remaining $mn$ elements are split into $m$ groups of $n$ with equal sums. Thus they provide a counterexample in characteristic $p$ of the statement proved in Theorem 1 for characteristic 0.

Proof Details- see paper
Example. This is a practical, hands-on way to construct the counterexample for small values of the parameters. Let $m = 3$, $n = 4$. Then $p = mn + 1 = 13$. As a primitive root modulo 13, we take $g = 2$.

Let us list the elements of $F_{13}^*$ as powers of the primitive root 2:

<table>
<thead>
<tr>
<th>$K$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^k \mod 13$</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>11</td>
<td>9</td>
<td>5</td>
<td>10</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

To get our 3 blocks of 4 upon the removal of 0, we take every third element in the row of the powers (second row) of the above table, so the three groups are

$$\{2, 3, 11, 10\}, \{4, 6, 9, 7\}, \text{ and } \{8, 12, 5, 1\}.$$  (7)
Thus if we remove the element 0 out of $F_{13}$, then the remaining 12 elements can be split into the above 3 groups of 4 with equal sums (the equal sums are zero in this case). What about removing other elements? Say, for example, we remove $2 \in F_{13}$. As indicated in the proof of Theorem 2, we add 2 to each element of a block in (7). Therefore, if we remove the element 2 out of $F_{13}$, then the remaining 12 elements can be split into the following 3 groups of size 4 with equal sums:

{4, 5, 0, 12}, {6, 8, 11, 9}, and {10, 1, 7, 3}.

As it is easy to see, the sum in each group is 8, which is consistent with the previous construction given in Theorem 2 ($na = 4 \cdot 2 = 8$). Table 1 below displays an interesting combinatorial pattern, showing all partitions in 3 groups of size 4 occurring when each element of $F_{13}$ is removed.
<table>
<thead>
<tr>
<th>Element removed</th>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
<th>Shift</th>
<th>Sum per group</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2, 3, 10, 11</td>
<td>4, 6, 7, 9</td>
<td>1, 5, 8, 12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3, 4, 11, 12</td>
<td>5, 7, 8, 10</td>
<td>0, 2, 6, 9</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0, 4, 5, 12</td>
<td>6, 8, 9, 11</td>
<td>1, 3, 7, 10</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>0, 1, 5, 6</td>
<td>7, 9, 10, 12</td>
<td>2, 4, 8, 11</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>1, 2, 6, 7</td>
<td>0, 8, 10, 11</td>
<td>3, 5, 9, 12</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2, 3, 7, 8</td>
<td>1, 9, 11, 12</td>
<td>0, 4, 6, 10</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>3, 4, 8, 9</td>
<td>0, 2, 10, 12</td>
<td>1, 5, 7, 11</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>4, 5, 9, 10</td>
<td>0, 1, 3, 11</td>
<td>2, 6, 8, 12</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>5, 6, 10, 11</td>
<td>1, 2, 4, 12</td>
<td>0, 3, 7, 9</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>6, 7, 11, 12</td>
<td>0, 2, 3, 5</td>
<td>1, 4, 8, 10</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>0, 7, 8, 12</td>
<td>1, 3, 4, 6</td>
<td>2, 5, 9, 11</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>0, 1, 8, 9</td>
<td>2, 4, 5, 7</td>
<td>3, 6, 10, 12</td>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>1, 2, 9, 10</td>
<td>3, 5, 6, 8</td>
<td>0, 4, 7, 11</td>
<td>12</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 1 shows that if any element of $F_{13}$ is removed, then the remaining 12 can be split into 3 groups of 4 with equal sums.
Main reference:

Mihai Caragiu and Rachael Harbaugh

Extending a Putnam Problem To Fields of Various Characteristics

JP Journal of Algebra, Number Theory and Applications
Vol 59, 33 - 45 (November 2022)
Works cited in the main reference


