Packing edge-colorings of graphs with maximum degree at most 4

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Grand Valley State University 2022 REU

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Definition: Proper Edge-Coloring

A **Proper j-Edge-Coloring** of a graph G is an assignment of the colors $\{1_1, 1_2, ..., 1_j\}$ to each edge in G such that

- incident edges receive distinct colors

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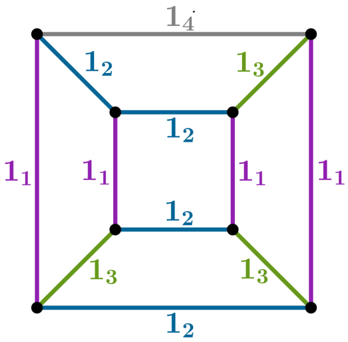

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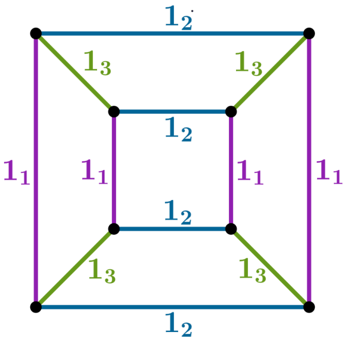

Definition: Strong Edge-Coloring

A **Strong k-Edge-Coloring** of a graph G is an assignment of colors $\{2_1, 2_2, ..., 2_k\}$ to each edge in G such that

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- any two edges that are incident to a common third edge receive distinct colors

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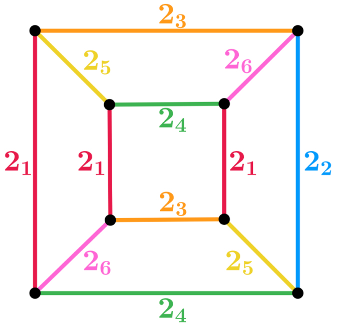

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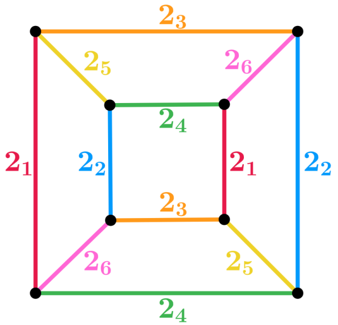

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Erdős-Nešetřil Conjecture (1985)

If a graph has maximum degree Δ then it has a strong edge-coloring with at most

- $\frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$ (when Δ is odd)
- $\frac{5}{4}\Delta^2$ (when Δ is even)

colors.

Definition: Packing Edge-Coloring

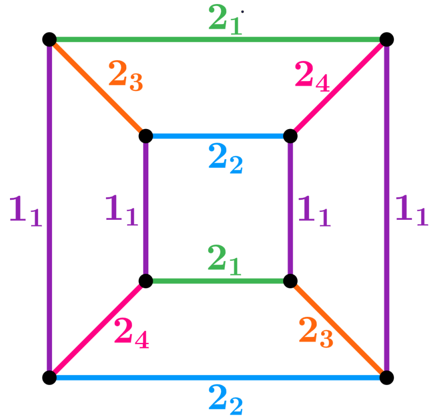
A $\mathbf{(1^j, 2^k)}$**-Packing Edge-Coloring** of a graph G is an assignment of the colors $\{1_1, 1_2, ..., 1_j\}$ and $\{2_1, 2_2, ..., 2_k\}$ to the edges in G such that

- incident edges receive distinct colors
- any two edges colored 2_i for the same $1 \leq i \leq k$ are not incident to a common edge.

Simply put:

- "1 colors" behave like proper colors
- "2 colors" behave like strong colors

$(1^1, 2^4)$-Packing Edge-Coloring Example



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Let G be a graph with $\Delta(G) = 4$

- The conjecture posits a strong 20-edge-coloring of G exist.
- It has been proven that a strong 21-edge-coloring of G exists.
- This guarantees the existence of a $(1^1, 2^{20})$-packing edge-coloring of G.

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Can we guarantee a $(1^1, 2^{19})$-packing edge-coloring of G?

Main Result

Every graph G with $\Delta \leq 4$ has a $(1^1, 2^{19})$-packing edge-coloring such that the 1-color only appears on edges whose endpoints are both vertices of degree 4.

- Implies the weaker claim that if $\Delta(G) \leq 4$, G has a $(1^1, 2^{19})$-packing edge-coloring.

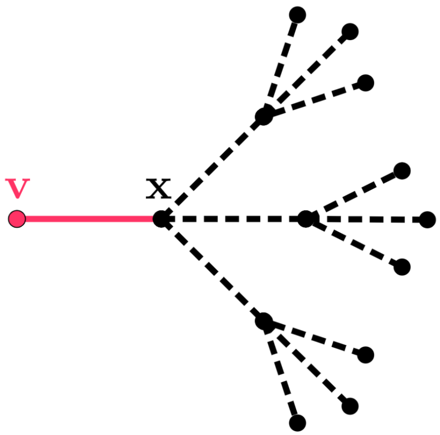
Minimal Counterexample Proof Structure

- For a contradiction, assume H is a smallest counterexample to the main result.

- We show H has no vertices of degree 1, 2, or 3 and no cycles of length 3, 4, or 5.

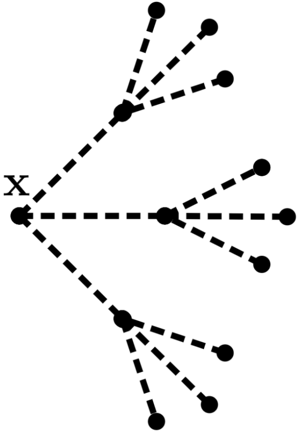
- Finally we use a modified Greedy Algorithm to show H has $(1^1, 2^{19})$-packing edge-coloring with the condition on the placement of the 1-color, so no such counterexample can exist.

H has no vertices of degree 1



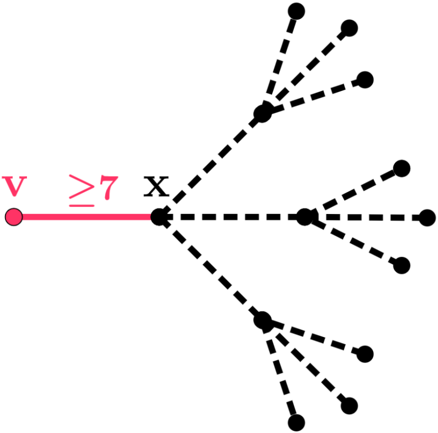
H has no vertices of degree 1

- Delete the vertex v to form a new graph H'.

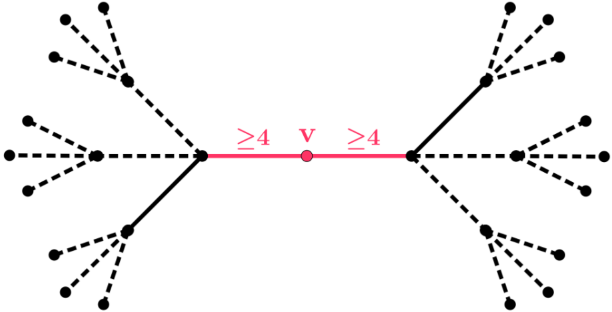


H has no vertices of degree 1

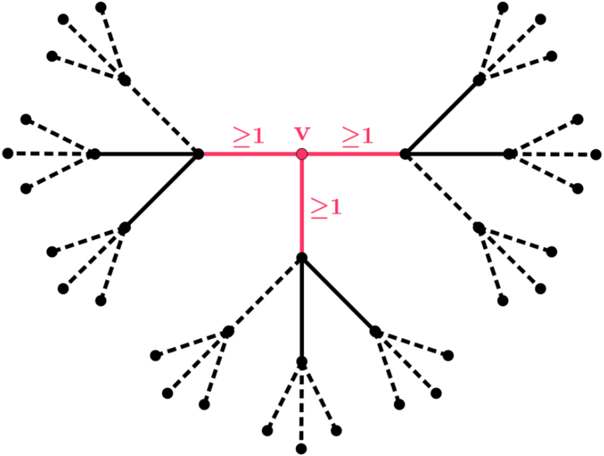
- Now, add v back in to the graph.



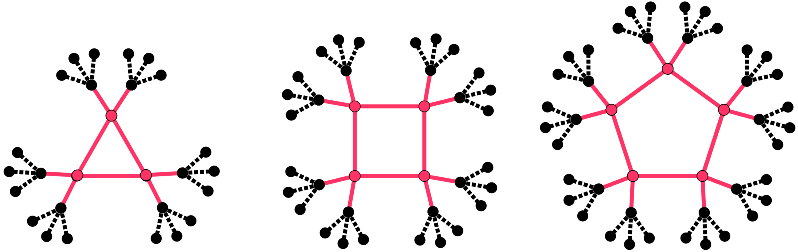
H has no vertices of degree 2



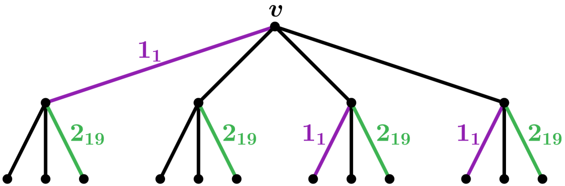
H has no vertices of degree 3

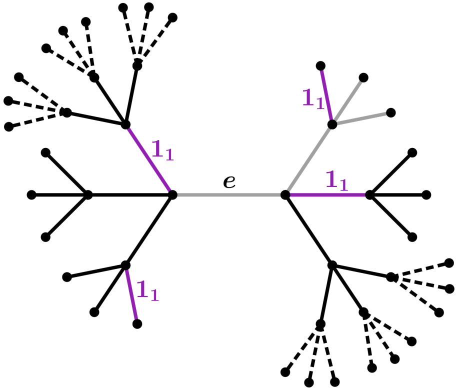


H has no cycles of length 3, 4, or 5



Finishing the Proof





Other Results and Possible Questions

- We have also proved that every graph with $\Delta \leq 4$ has a $(1^2, 2^{17})$-packing edge-coloring such that a 1-color only appears on edges whose endpoints are both vertices of degree 4.

- We are in the process of showing that every class 1 graph with $\Delta \leq 4$ has a $(1^3, 2^6)$-packing edge-coloring.

- Additional Question: What is the smallest k such that every graph with $\Delta \leq 4$ is has a $(1^1, 2^k)$-packing edge-coloring?

Thank You!

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References

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