Computational Geometry and Archeological Refits

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Introduction

Summer research at **Carleton College** with **Prof. Rob Thompson** (and fellow student **Evelene Zhang)**

Prototyping automatic bone refitting based on existing mathematical ideas

Refitting (and why it matters)

Refitting is the **reassembly of bone/stone fragments**.

Allows for a better understanding of early human life.

Impractical due to funding and time

Automatic refitting

Make 3D scans of bone fragments

Input 3D scans of bones

Analyze the geometry of bones

Find best alignment of the bones

Apply motions to achieve this alignment

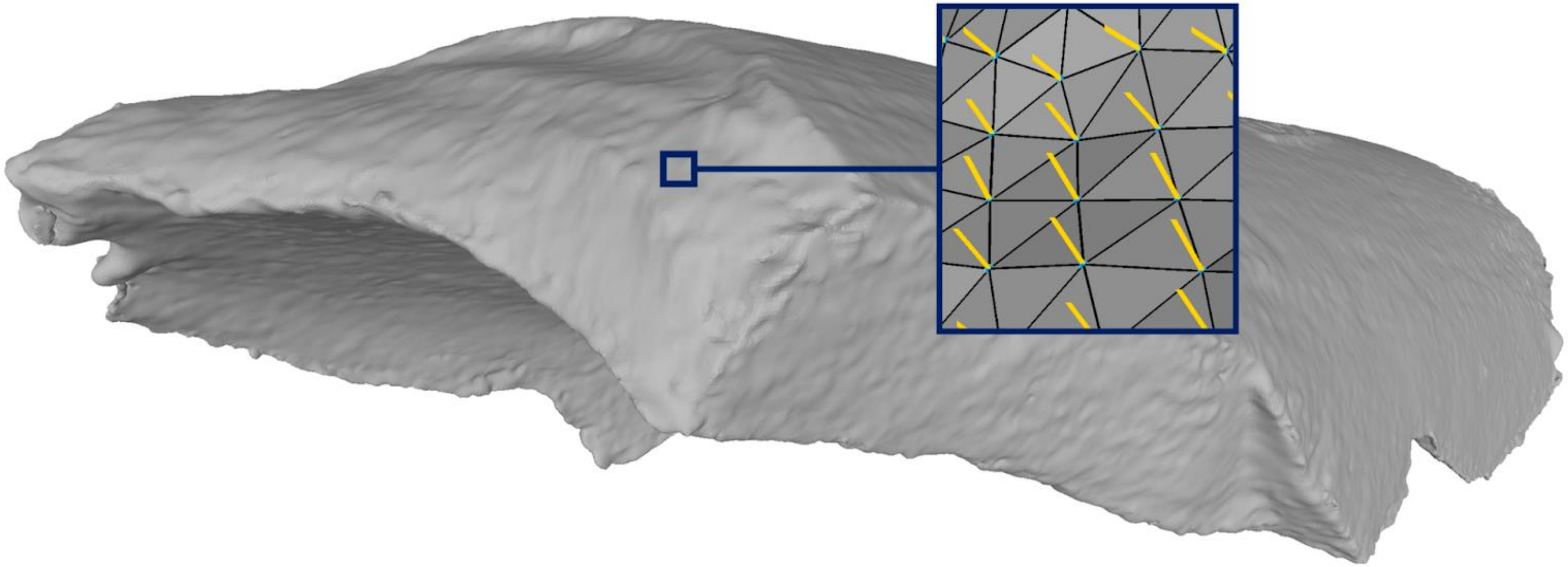
Output aligned bones!

Our input data

Provided by the **AMAAZE Consortium at the University of Minnesota**

.ply “point cloud” file format

Vertices
Edges
Normal vectors
Triangular faces



Our input data

“Archaeological” faces

Only **break faces** are used in our algorithm

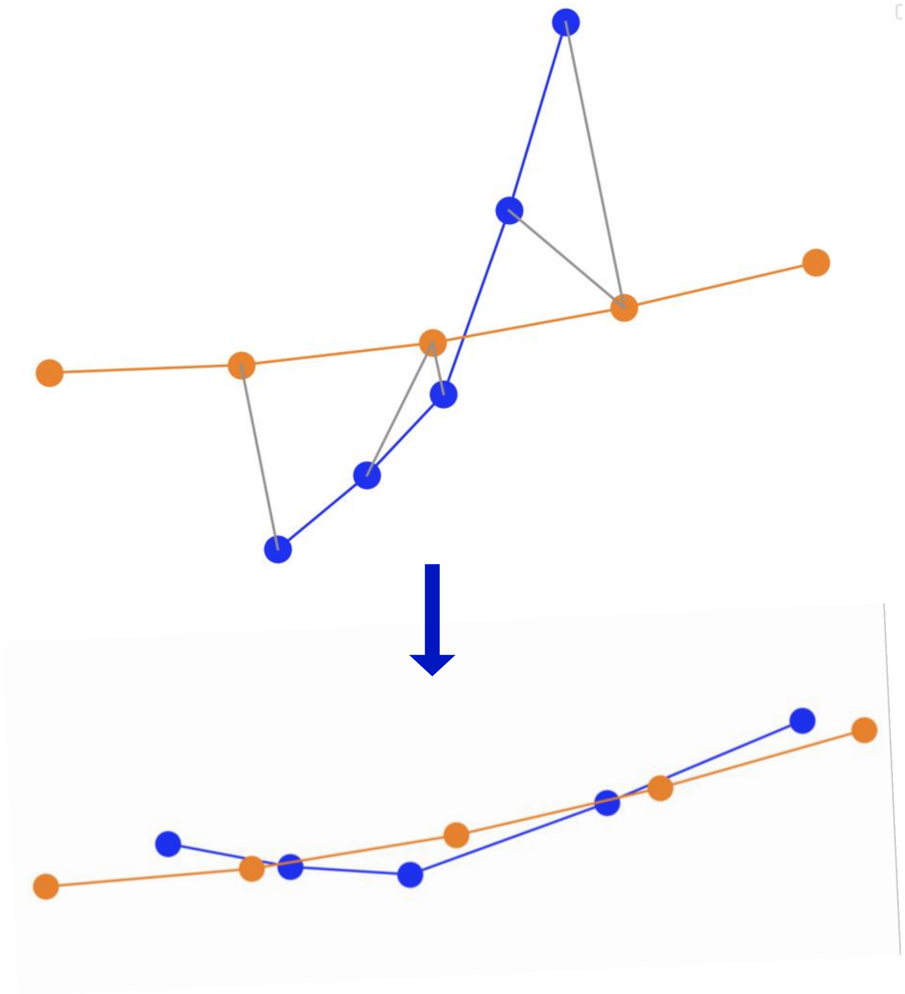
Iterative Closest Point (ICP)

Let one shape be **stationary** and let the other be **moving**.

Pair points.

Find the **minimizing transformation**.

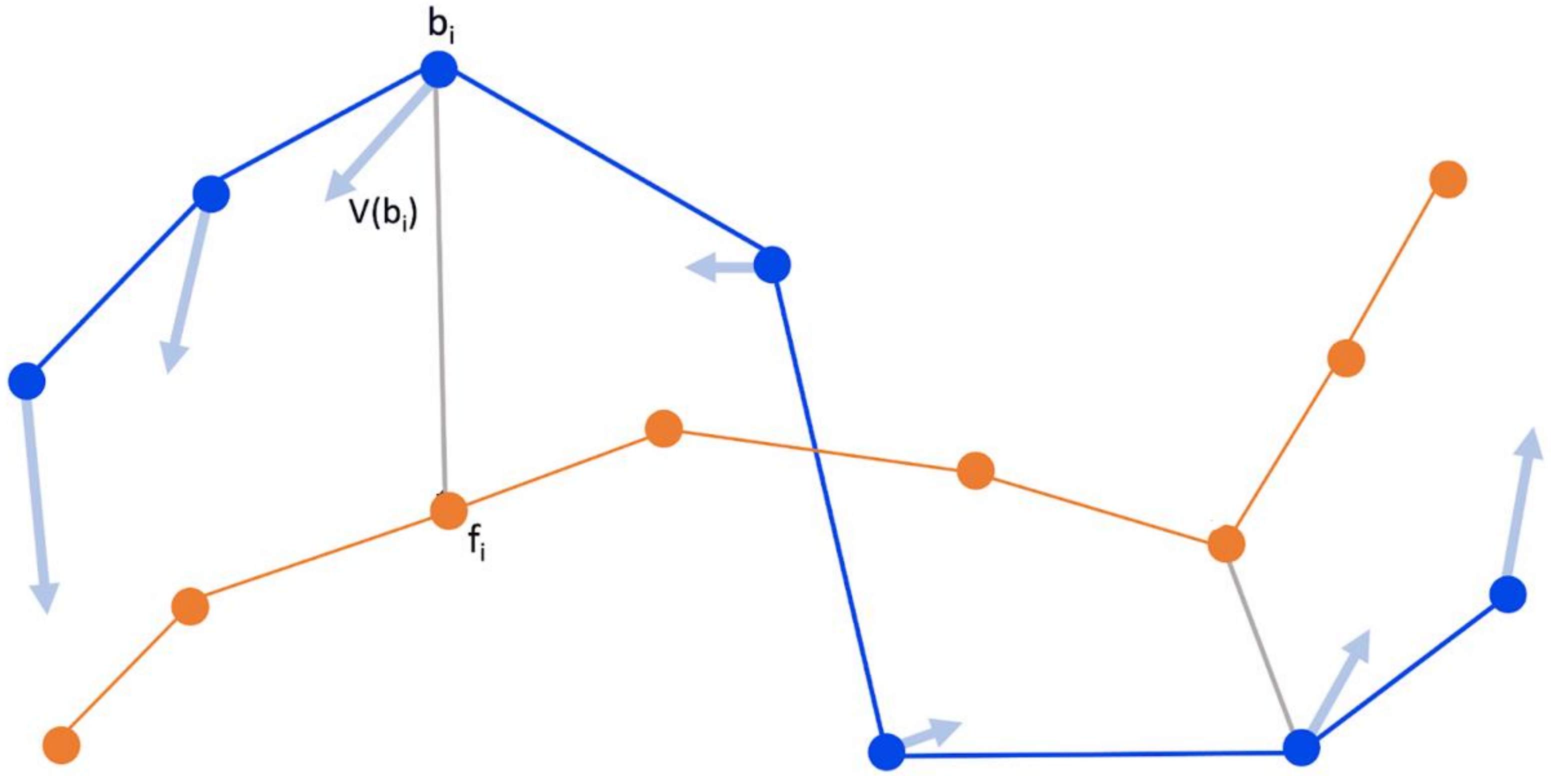
Iterate!



Iterative Closest Point (ICP)

Motion found using an optimization problem:

$$\min_{(\mathbf{c},\bar{\mathbf{c}})} \sum_{\mathbf{i}=1}^{\mathbf{n_B}} ||\mathbf{b_i} + \mathbf{v}(\mathbf{b_i}) - \mathbf{f_i}||^2 \qquad \mathbf{v}(\mathbf{b_i}) = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{b_i}$$

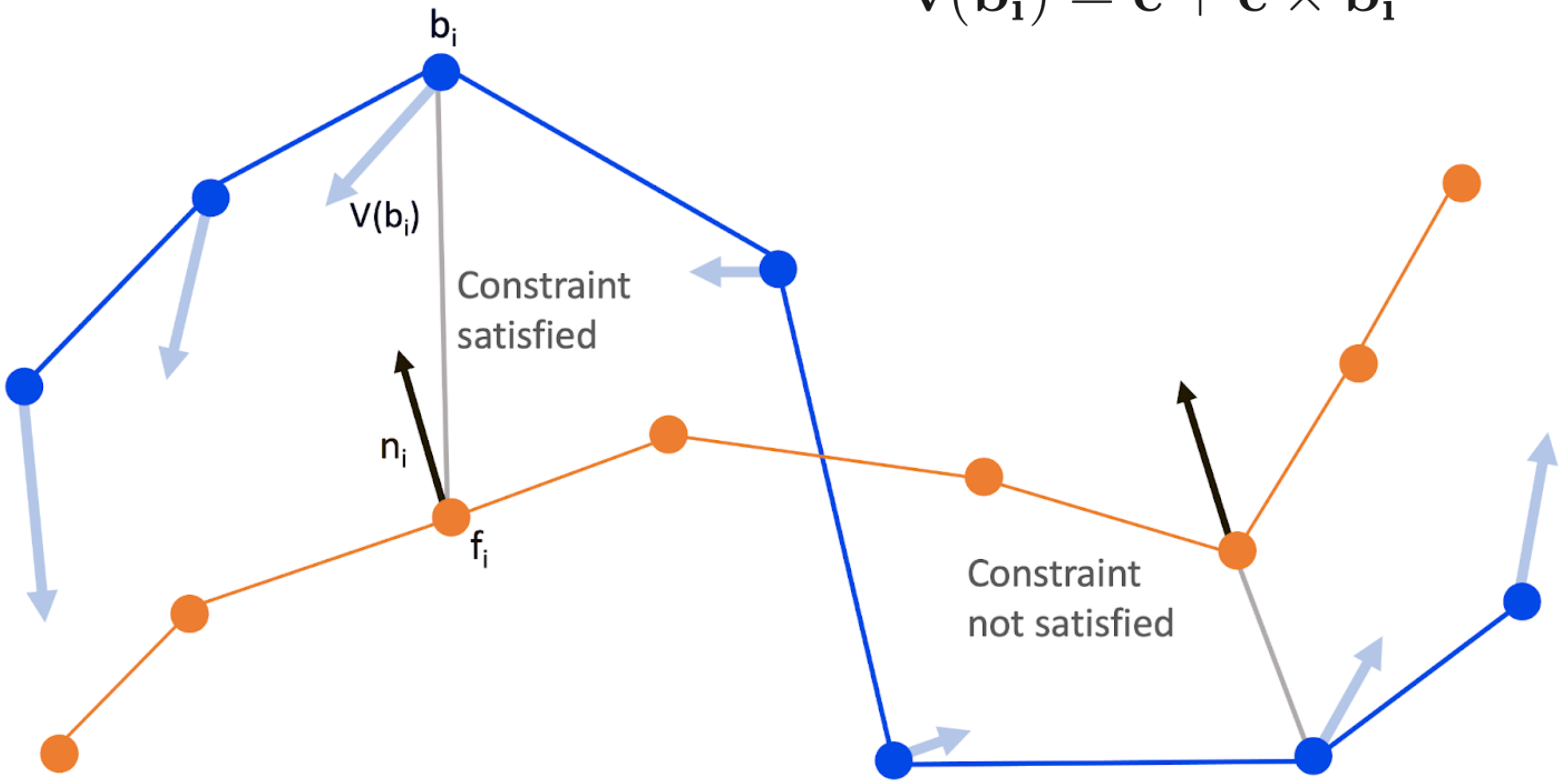


Overlap Problem

Solving the overlap problem

$$\min_{(\mathbf{c},\bar{\mathbf{c}})} \sum_{\mathbf{i}=1}^{\mathbf{n_B}} ||\mathbf{b_i} + \mathbf{v}(\mathbf{b_i}) - \mathbf{f_i}||^2 \ \mathbf{s.t.}\ \mathbf{n_i^T} \cdot (\mathbf{b_i} + \mathbf{v}(\mathbf{b_i}) - \mathbf{f_i}) > \mathbf{0}, \forall \mathbf{i}$$

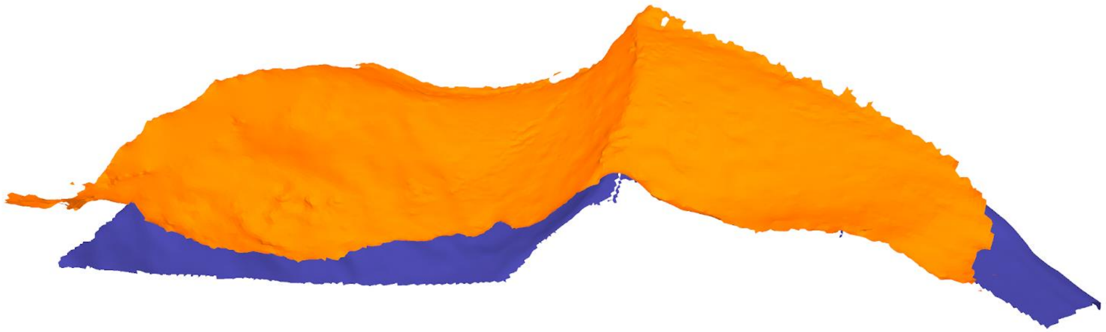
$$\mathbf{v}(\mathbf{b_i}) = \bar{\mathbf{c}} + \mathbf{c} \times \mathbf{b_i}$$



Overlap problem solved!

Without constraint

With constraint

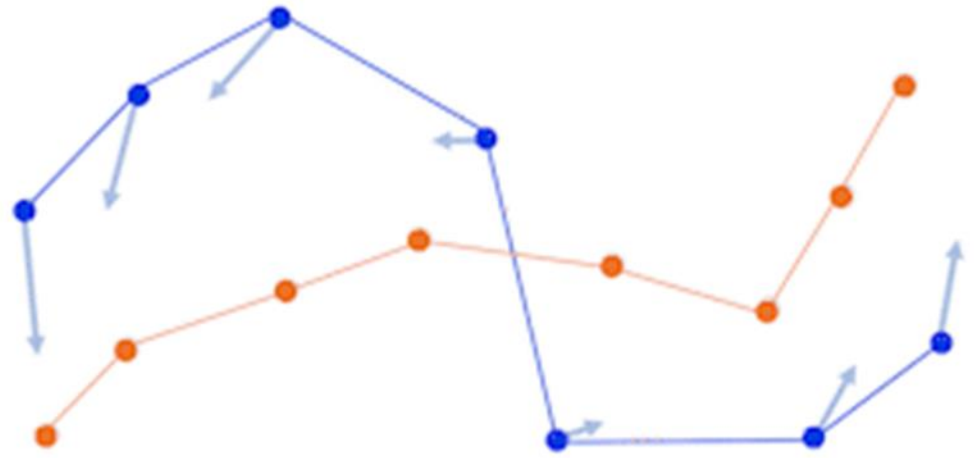


Finding the rigid motion

Any rigid transformation is given by a uniform helical motion.

Our optimization finds a **velocity vector field**:

$$\mathbf{v}(\mathbf{b_i}) = \overline{\mathbf{c}} + \mathbf{c} \times \mathbf{b_i}$$



We “integrate” to find a rigid motion.

Finding a helical motion

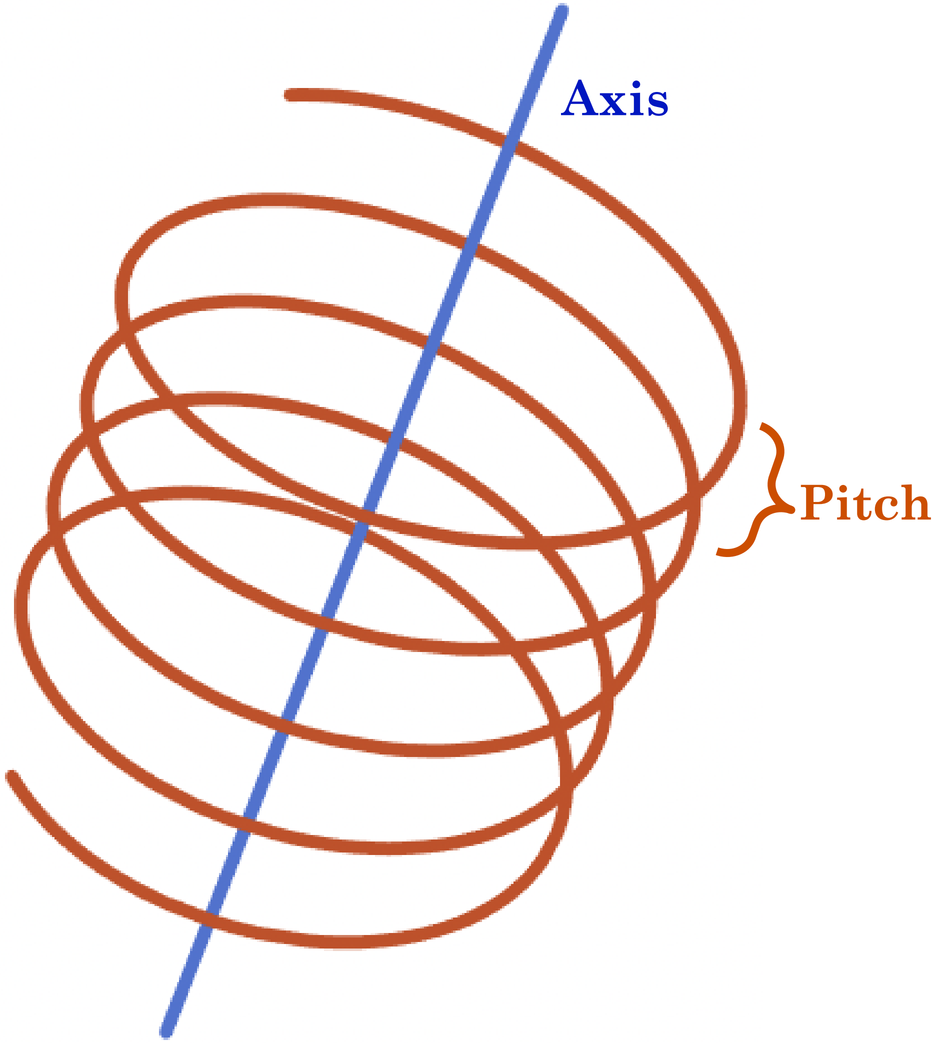
Direction vector: $\mathbf{c}$

Distance between turns of the helix: $\mathbf{p} = \frac{\mathbf{c}\cdot\bar{\mathbf{c}}}{\mathbf{c}\cdot\mathbf{c}}$

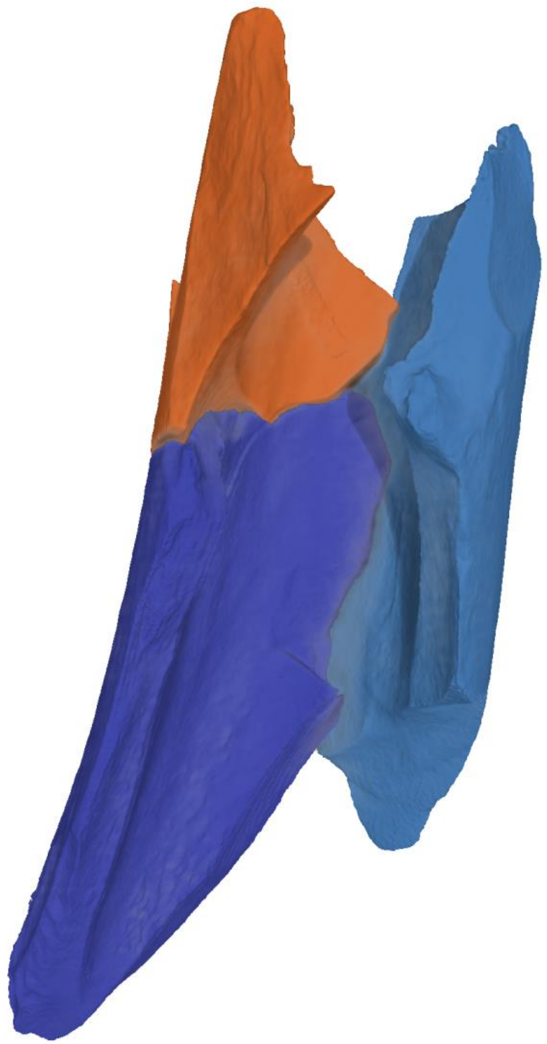
Uniform helical motion around the z axis:

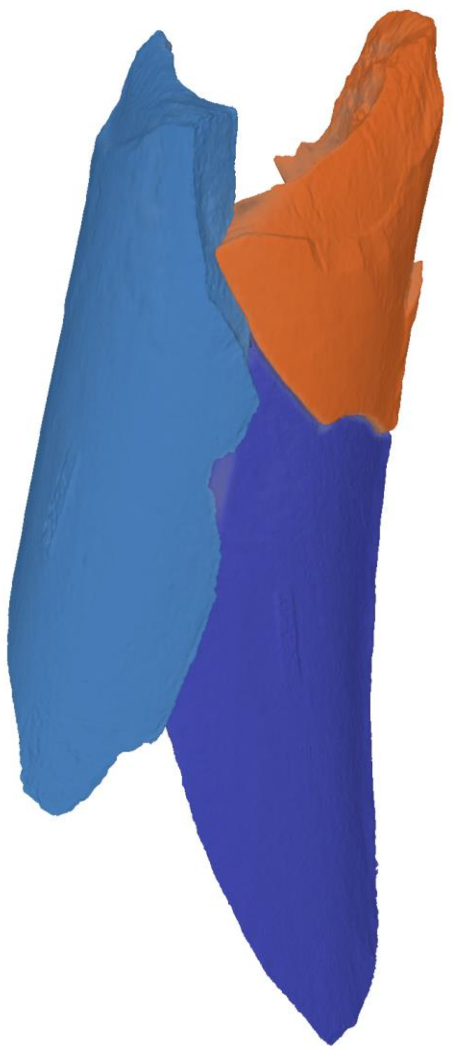
$$\mathbf{x} \mapsto \mathbf{x}(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ p \cdot t \end{bmatrix}$$

We **transform, apply motion, transform back**.



Results





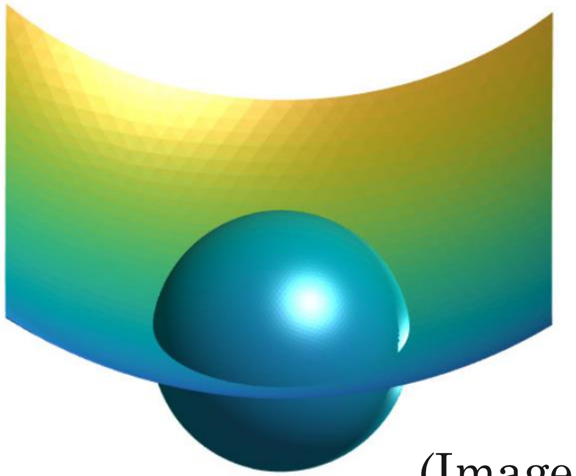
Ongoing Work

Multipiece refits:

$$\min_{(\bar{\mathbf{c}}_1,\mathbf{c}_1,\ldots,\bar{\mathbf{c}}_n,\mathbf{c}_n)} \sum_{(j,k)\in\mathcal{A}} \sum_{\mathbf{b}\in A_j} \left[\left(\mathbf{b}+\mathbf{v}_{jk}(\mathbf{b})-\mathbf{f}_b^k\right)^T \cdot \mathbf{n}_b^k\right]^2$$

$$\text{subject to} \quad \left(\mathbf{b}+\mathbf{v}_{jk}(\mathbf{b})-\mathbf{f}_b^k\right)^T \cdot \mathbf{n}_b^k > 0 \qquad \forall (j,k)\in\mathcal{A},\ \mathbf{b}\in A_j.$$

Spherical volume invariant:



(Image from O'Neill et al.)

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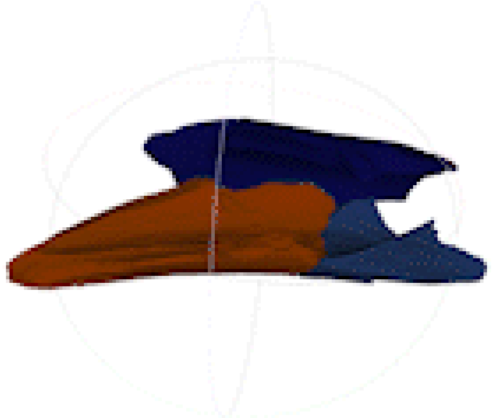

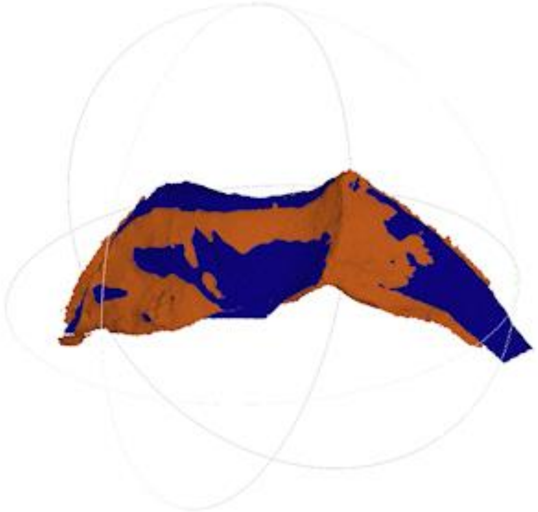

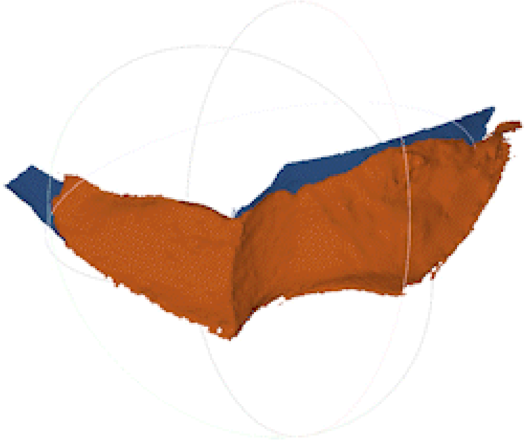

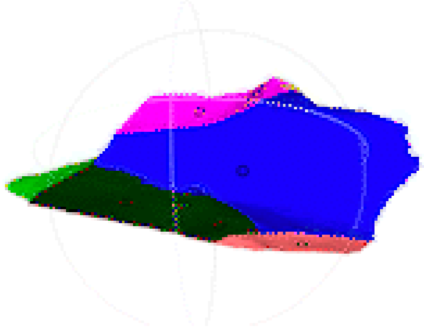

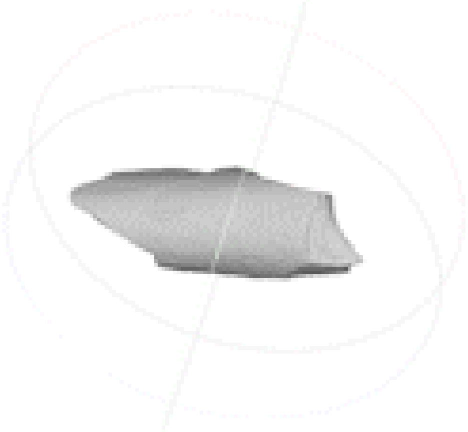

Thank you!

Any questions?