Complementation of Subquandles

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Definition

A **quandle** is a set $Q$ equipped with binary operations $\triangleright$ and $\triangleright^{-1}$ satisfying the following for all $x, y, z \in Q$: (1)
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Definition

A quandle is a set $Q$ equipped with binary operations $\triangleright$ and $\triangleright^{-1}$ satisfying the following for all $x, y, z \in Q$:

1. **Idempotence:** $x \triangleright x = x$,
2. **Inversion:** $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$, 
3. **Distributivity:** $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$. 

$Q' \subseteq Q$ is a subquandle of $Q$ if it is closed under $\triangleright$ and $\triangleright^{-1}$. Subquandles are denoted as $Q' \trianglelefteq Q$ or $Q' \vartriangleleft Q$. 
Preliminary Definitions

Definition

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Q2. Inversion: $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$,

Q3. Distributivity: $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.
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$Q' \subseteq Q$ is a **subquandle** of $Q$ if it closed under $\triangleright$ and $\triangleright^{-1}$. Subquandles are denoted as $Q' \leq Q$ or $Q' \prec Q$. 
Example

The **Tait quandle** \((T_3, \triangleright)\) with underlying set \(\{1, 2, 3\}\) has the following multiplication table:

<table>
<thead>
<tr>
<th>(\triangleright)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
Example

Let \( G \) be a group. \( \text{Conj}(G) \) is called the **conjugacy quandle** of a group, constructed by elements in \( G \) with the binary operation being defined for all \( g, h \in G \) as:

\[
g \triangleright h = g^{-1}hg.
\]
Example

Let $A$ be an abelian group, and let us define $x \triangleright_{\text{dih}} y = 2y - x$ for any $x, y \in A$. Then $(A, \triangleright_{\text{dih}})$ forms a quandle called the Takasaki quandle, $T(A)$.

The operation $\triangleright_{\text{dih}}$ of $T(A)$ is called the dihedral action.

Note that for $T(A)$, $(x \triangleright_{\text{dih}} y) \triangleright_{\text{dih}} y = x$ for all $x, y \in A$. This gives that $\triangleright_{\text{dih}} = \triangleright^{-1}_{\text{dih}}$, and that $T(A)$ is associative.
Motivations

- Knot theory!
- Quandles encode a complete invariant for classical knots, up to orientation reversal.
- Each quandle axiom corresponds to each Reidemeister move.
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Set-Theoretic Complementation

Definition

A subquandle $Q' \leq Q$ is strongly complemented if $Q \setminus Q' \leq Q$. 
Set-Theoretic Complementation

**Definition**

A subquandle $Q' \precsim Q$ is **strongly complemented** if $Q \setminus Q' \precsim Q$.

**Example**

In the quandle $Q$ represented by the matrix

$$M_Q = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix},$$

the subquandles

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, [1]$$

are strongly complemented.
Complete Classification of Strong Complementation

Theorem

Let \( Q \) be a quandle, and let \( Q' \preceq Q \). Denote the subquandle lattice of \( Q \) by \( \mathcal{L}(Q) \). The following are equivalent:

- \( Q \backslash Q' \preceq Q \),
- \( Q' \) is a union of orbits under the action of \( \text{Inn}(Q) \) on \( Q \),
- \( Q' \) is a fixed point of the action of \( \text{Inn}(Q) \) on \( \mathcal{L}(Q) \),
- \( Q = #(Q', Q \backslash Q', M) \) for a mesh \( M \) as constructed in the Orbit Decomposition Theorem of Ehrman et al. (2).
Complete Classification of Strong Complementation

**Theorem**

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- $Q \setminus Q' \preceq Q$,
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Definition

The set of subquandles of any quandle \( Q \) under inclusion forms a lattice \((3)\), which we denote as \( \mathcal{L}(Q) \).

Definition

Given two subquandles \( Q_1, Q_2 \preceq Q \), their meet is \( Q_1 \land Q_2 = Q_1 \cap Q_2 \) and their join is \( Q_1 \lor Q_2 = \langle \langle Q_1 \cup Q_2 \rangle \rangle \).
Definition
The set of subquandles of any quandle $Q$ under inclusion forms a lattice (3), which we denote as $\mathcal{L}(Q)$.

Definition
Given two subquandles $Q_1, Q_2 \leq Q$, their meet is $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ and their join is $Q_1 \vee Q_2 = \langle\langle Q_1 \cup Q_2\rangle\rangle$.

Definition (Complemented Subquandle Lattices)
$Q_1 \leq Q$ is complemented in $Q$ if there is some $Q_2 \leq Q$ such that $Q_1 \wedge Q_2 = \emptyset$, and $Q_1 \vee Q_2 = Q$. The subquandle lattice $\mathcal{L}(Q)$ is complemented if every subquandle is complemented.
Complemented Sublattice Examples

Example

All finite quandles have a complemented subquandle lattice. \[(3)\]
Complemented Sublattice Examples

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All finite quandles have a complemented subquandle lattice. [(3)

Example
\((\mathbb{Q}, \triangleright_{\text{dih}})\) does not have a complemented subquandle lattice. In particular, \(\{0\}\) has no complement. (3)
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\((\mathbb{Q}, \triangleright_{\text{dih}})\) does not have a complemented subquandle lattice. In particular, \(\{0\}\) has no complement. (3)

Example
\(\mathbb{Q}/\mathbb{Z}\) does not have a complemented subquandle lattice. In particular, \(\{\mathbb{Z}\}\) has no complement.
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References


Thank you for listening!

Questions?

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Additional Slides
Definitions for Group Action of $\text{Inn}(Q)$

**Definition**

If $(Q, \triangleright)$ and $(R, \triangleright_1)$ are quandles, a **quandle homomorphism** $f : Q \to R$ is a function satisfying $f(a \triangleright b) = f(a) \triangleright_1 f(b)$ for every $a, b \in Q$. A bijective quandle homomorphism is a **quandle isomorphism**. A quandle isomorphism with equal domain and codomain is a **quandle automorphism**. The quandle automorphisms form the **automorphism group** $\text{Aut}(Q)$.

**Definition**

Given a quandle $Q$ and an element $y \in Q$, the **symmetry at $y$** is the automorphism of $Q$ of the form $S_y : x \mapsto x \triangleright y$.

The **inner automorphism group** of $Q$ is defined as $\text{Inn}(Q) = \langle \{ S_q \mid q \in Q \} \rangle$. Note that $\text{Inn}(Q) \trianglelefteq \text{Aut}(Q)$.
Definitions for Group Action of Inn($Q$)

Definition

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Definitions for Group Action of Inn\((Q)\)

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Definitions for Group Action of \( \text{Inn}(Q) \)

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Given a quandle \( Q \) and an element \( y \in Q \), the **symmetry** at \( y \) is the automorphism of \( Q \) of the form \( S_y : x \mapsto x \triangleright y \).

The **inner automorphism group** of \( Q \) is defined \( \text{Inn}(Q) = \langle \{ S_q \mid q \in Q \} \rangle \). Note that \( \text{Inn}(Q) \leq \text{Aut}(Q) \) (1).
The action of $\text{Inn}(Q)$ on $Q$ by functional application.

This action allows us to construct an action of $\text{Inn}(Q)$ upon $\mathcal{L}(Q)$.

**Definition**

The action of $\text{Inn}(Q)$ on $Q'$ is also given by functional application, denoted $Q' \cdot \text{Inn}(Q)$.

The action of $\text{Inn}(Q)$ upon $\mathcal{L}(Q)$ is defined $Q'\sigma = \sigma(Q')$ for all $Q' \in \mathcal{L}(Q)$, and for all $\sigma \in \text{Inn}(Q)$. The orbit of $Q'$ under this action is denoted by $\left[Q'\right] \cdot \text{Inn}(Q) = \{Q'\sigma : \sigma \in \text{Inn}(Q)\}$. 
Definitions for Orbit Decomposition

Definition (Orbit)
The orbit of an element $s \in Q$ is the subset of elements $t \in Q$ such that there exists some $p \in \text{Inn}(Q)$ where $p$ maps $s$ to $t$. 
**Semidisjoint Union**

**Definition (Ehrman et al.)**

Given a sequence of quandles $Q_1, \ldots, Q_n$ and a $n \times n$ matrix of group homomorphisms $(M)_{ij} = g_{ij}$, Ehrman et al. (2) defined the semidisjoint union as follows:

\[
\#(Q_1, \ldots, Q_n, M) = \left( \bigsqcup_{i=1}^{n} Q_i, \triangleright \right).
\]
Semidisjoint Union

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Given a sequence of quandles \(Q_1, \ldots, Q_n\) and a \(nxn\) matrix of group homomorphisms \((M)_{ij} = g_{ij}\), Ehrman et al. (2) defined the semidisjoint union as follows:

\[
\#(Q_1, \ldots, Q_n, M) = \left( \bigcup_{i=1}^{n} Q_i, \triangleright \right).
\]

- Each entry of the matrix \(g_{ij} : \text{Adconj}(Q_i) \rightarrow \text{Aut}(Q_j)\) is a group homomorphism.
- \(\triangleright\) is defined as \(x \triangleright y = x \cdot g_{ij}(|y|_{Q_i})\) for \(x \in Q_i\) and \(y \in Q_j\).
- Note that we are not guaranteed that the semidisjoint union is a quandle. If the matrix \(M\) gives rise to a quandle, it is called a mesh.
- Ehrman et al. provided a necessary and sufficient condition for \(M\) to be a mesh.
Theorem (Ehrman et al.)

Let \( Q \) be a quandle, and let \( Q_1, \ldots, Q_n \) be its orbits under the inner automorphism group. Then we can construct a mesh \( M \) such that

\[
Q = \#(Q_1, \ldots, Q_n, M).
\]

- Note that the orbits need not be connected. Hence the orbits themselves may be decomposable via the previous theorem.
Theorem

Suppose $Q$ is a quandle, with subquandles $Q'' \leq Q' \leq Q$, such that $Q''$ is strongly complemented within $Q'$, while $Q'$ is strongly complemented within $Q$. Then $Q''$ is complemented within $Q$ by the subquandle $Q \setminus Q'' \cdot \text{Inn}(Q)$.
We consider the dual to ind-finite quandles:

A quandle $Q$ is **profinite** if it is the inverse limit of an inverse system composed of a family of finite quandles and their morphisms.

We proved profinite quandles are quandles under coordinatewise operations.

We proved profinite abelian groups are profinite Takasaki quandles $(x \triangleright y = 2y - x)$ under coordinatewise operations.

Are the sublattices of profinite quandles complemented? Or does there exist a non-complemented profinite quandle?
Correspondence Between Quandle Axioms and Reidemeister Moves

R1

\[ \begin{align*}
\text{idempotence} \\
\end{align*} \]

R2

\[ \begin{align*}
\text{right-inversive} \\
\end{align*} \]

R3

\[ \begin{align*}
\text{right-distributive} \\
\end{align*} \]