Complementation of Subquandles

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Preliminary Definitions

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$Q' \subseteq Q$ is a **subquandle** of Q if it closed under $\triangleright$ and $\triangleright^{-1}$.
Subquandles are denoted as $Q' \preccurlyeq Q$ or $Q' \prec Q$.

Preliminary Examples

Example

The **Tait quandle** $(\mathbf{T}_3, \triangleright)$ with underlying set $\{1, 2, 3\}$ has the following multiplication table:

$\triangleright$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Preliminary Examples

Example

Let G be a group. $\mathrm{Conj}(G)$ is called the **conjugacy quandl**e of a group, constructed by elements in G with the binary operation being defined for all $g, h \in G$ as:

$$g \triangleright h = g^{-1}hg.$$

Preliminary Examples

Example

Let A be an abelian group, and let us define $x \rhd_{\mathrm{dih}} y = 2y - x$ for any $x, y \in A$. Then $(A, \rhd_{\mathrm{dih}})$ forms a quandle called the **Takasaki quandle**, $T(A)$.

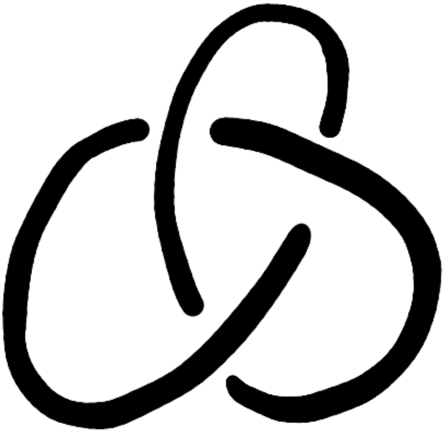
The operation $\rhd_{\mathrm{dih}}$ of $T(A)$ is called the *dihedral action*.

Note that for $T(A)$, $(x \rhd_{\mathrm{dih}} y) \rhd_{\mathrm{dih}} y = x$ for all $x, y \in A$. This gives that $\rhd_{\mathrm{dih}} = \rhd_{\mathrm{dih}}^{-1}$, and that $T(A)$ is associative.

Motivations

- Knot theory!
- Quandles encode a complete invariant for classical knots, up to orientation reversal.
- Each quandle axiom corresponds to each Reidemeister move.

Motivations



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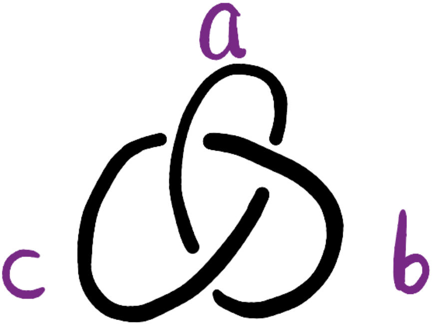

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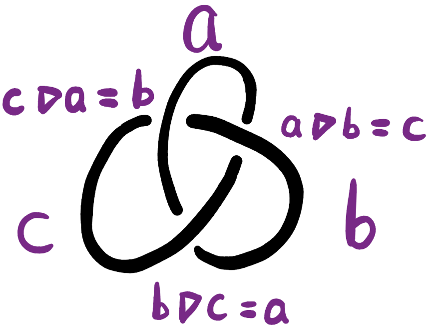

Set-Theoretic Complementation

Definition

A subquandle $Q' \preccurlyeq Q$ is **strongly complemented** if $Q \setminus Q' \preccurlyeq Q$.

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Example

In the quandle Q represented by the matrix

$$M_Q = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix},$$

the subquandles

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}, [1]$$

are strongly complemented.

Complete Classification of Strong Complementation

Theorem

Let Q be a quandle, and let $Q' \preccurlyeq Q$. Denote the subquandle lattice of Q by $\mathcal{L}(Q)$. The following are equivalent:

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Let Q be a quandle, and let $Q' \preccurlyeq Q$. Denote the subquandle lattice of Q by $\mathcal{L}(Q)$. The following are equivalent:

- $Q \setminus Q' \preccurlyeq Q$,
- Q' *is a union of orbits under the action of* $\mathrm{Inn}(Q)$ *on* Q,
- Q' *is a fixed point of the action of* $\mathrm{Inn}(Q)$ *on* $\mathcal{L}(Q)$,
- $Q = \#(Q', Q \setminus Q', M)$ *for a mesh* M *as constructed in the Orbit Decomposition Theorem of Ehrman et al. (2)*

Subquandle Lattice Complementation

Definition

The set of subquandles of any quandle Q under inclusion forms a **lattice** (3), which we denote as $\mathcal{L}(Q)$.

Definition

Given two subquandles $Q_1, Q_2 \preccurlyeq Q$, their **meet** is $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ and their **join** is $Q_1 \vee Q_2 = \langle\langle Q_1 \cup Q_2 \rangle\rangle$.

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Definition (Complemented Subquandle Lattices)

$Q_1 \preccurlyeq Q$ is **complemented** in Q if there is some $Q_2 \preccurlyeq Q$ such that $Q_1 \wedge Q_2 = \emptyset$, and $Q_1 \vee Q_2 = Q$. The subquandle lattice $\mathcal{L}(Q)$ is complemented if every subquandle is complemented.

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All finite quandles have a complemented subquandle lattice. [(3)

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$(\mathbb{Q}, \triangleright_{\mathrm{dih}})$ does not have a complemented subquandle lattice. In particular, $\{0\}$ has no complement. (3)

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$(\mathbb{Q}, \rhd_{\mathrm{dih}})$ does not have a complemented subquandle lattice. In particular, $\{0\}$ has no complement. (3)

Example

$\mathbb{Q}/\mathbb{Z}$ does not have a complemented subquandle lattice. In particular, $\{\mathbb{Z}\}$ has no complement.

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Thank you for listening! Questions?

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Additional Slides

Definitions for Group Action of $\mathrm{Inn}(Q)$

Definition

- If $(Q, \triangleright)$ and $(R, \triangleright_1)$ are quandles, a **quandle homomorphism** $f : Q \to R$ is a function satisfying $f(a \triangleright b) = f(a) \triangleright_1 f(b)$ for every $a, b \in Q$.

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- A bijective quandle homomorphism is a **quandle isomorphism**.
- A quandle isomorphism with equal domain and codomain is a **quandle automorphism**.
- The quandle automorphisms form the **automorphism group**, $\mathrm{Aut}(Q)$.

Definition

Given a quandle Q and an element $y \in Q$, the **symmetry** at y is the automorphism of Q of the form $S_y : x \mapsto x \triangleright y$.

The **inner automorphism group** of Q is defined $\mathrm{Inn}(Q) = \langle \{S_q \mid q \in Q\} \rangle$. Note that $\mathrm{Inn}(Q) \trianglelefteq \mathrm{Aut}(Q)$ (1).

Action of $\mathrm{Inn}(Q)$ cont.

- $\mathrm{Inn}(Q)$ acts on Q by functional application.
- This action allows us to construct an action of $\mathrm{Inn}(Q)$ upon $\mathcal{L}(Q)$.

Definition

The action of $\mathrm{Inn}(Q)$ on Q' is also given by functional application, denoted $Q' \cdot \mathrm{Inn}(Q)$.

The action of $\mathrm{Inn}(Q)$ upon $\mathcal{L}(Q)$ is defined $Q'\sigma = \sigma(Q')$ for all $Q' \in \mathcal{L}(Q)$, and for all $\sigma \in \mathrm{Inn}(Q)$. The orbit of Q' under this action is denoted by $[Q'] \cdot \mathrm{Inn}(Q) = \{Q'\sigma : \sigma \in \mathrm{Inn}(Q)\}$.

Definitions for Orbit Decomposition

Definition (Orbit)

The orbit of an element $s \in Q$ is the subset of elements $t \in Q$ such that there exists some $p \in \mathrm{Inn}(Q)$ where p maps s to t.

Semidisjoint Union

Definition (Ehrman et al.)

Given a sequence of quandles $Q_1, \ldots, Q_n$ and a nxn matrix of group homomorphisms $(M)_{ij} = g_{ij}$:, Ehrman et al. (2) defined the **semidisjoint union** as follows:

$$\#(Q_1, \ldots, Q_n, M) = \Big(\coprod_{i=1}^{n} Q_i, \triangleright \Big).$$

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- Each entry of the matrix $g_{ij} : \mathrm{Adconj}(Q_i) \to \mathrm{Aut}(Q_j)$ is a group homomorphism.
- $\triangleright$ is defined as $x \triangleright y = x \cdot g_{ij}(|y|_{Q_i})$ for $x \in Q_i$ and $y \in Q_j$.
- Note that we are not guaranteed that the semidisjoint union is a quandle. If the matrix M gives rise to a quandle, it is called a **mesh**.
- Ehrman et al. provided a necessary and sufficient condition for M to be a mesh.

Orbit Decomposition Theorem

Theorem (Ehrman et al.)

Let Q be a quandle, and let $Q_1, \ldots, Q_n$ be its orbits under the inner automorphism group. Then we can construct a mesh M such that

$$Q = \#(Q_1, \ldots, Q_n, M).$$

- Note that the orbits need not be connected. Hence the orbits themselves may be decomposable via the previous theorem.

Partial Transitivity Criterion for Strong Complementedness

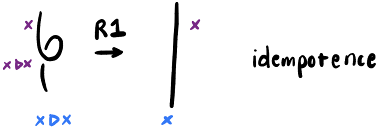
Theorem

Suppose Q is a quandle, with subquandles $Q'' \preccurlyeq Q' \preccurlyeq Q$, such that Q'' is strongly complemented within Q', while Q' is strongly complemented within Q. Then Q'' is complemented within Q by the subquandle $Q \setminus Q'' \cdot \mathrm{Inn}(Q)$.

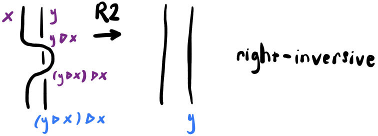
Future Work

- We consider the dual to ind-finite quandles:
- A quandle Q is **profinite** it is the inverse limit of an inverse system composed of a family of finite quandles and their morphisms.
- We proved profinite quandles are quandles under coordinatewise operations.
- We proved profinite abelian groups are profinite Takasaki quandles ($x \triangleright y = 2y - x$) under coordinatewise operations.
- Are the sublattices of profinite quandles complemented? Or does there exist a non-complemented profinite quandle?

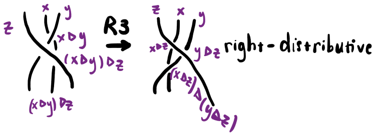
Correspondence Between Quandle Axioms and Reidemeister Moves



R1



R2



R3