# Computational Algebra

## by Polymath Jr.

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**21st January 2023**

# » Outline

## Introduction

* What is a Polynomial Ring?
* What is an Ideal?
* What is a Minimal Free Resolution?
* What are Betti Numbers?

## Introduction

* **What is a Polynomial Ring?**
* **What is an Ideal?**
* **What is a Minimal Free Resolution?**
* **What are Betti Numbers?**

# » What is a Polynomial?

**Definition (Monomial)**

A *monomial* is a product of variables and scalars.

**Example**

$3x^9y^2$ is a monomial, but $x^2 + z$ is not.

**Definition (Polynomial)**

A *polynomial* is a sum of monomials.

**Example**

$x^2 + y^2 + 2z^2$ and $6x^7 - 2xy^3$ are both polynomials.

# » What is a Polynomial Ring?

In the following slides, let $k$ denote a field (ex. $\mathbb{R}$, $\mathbb{Q}$, or the integers modulo 2, denoted here as $\mathbb{F}_2$)

**Definition (Polynomial Ring)**

A *polynomial ring* $R = k[x_1, \ldots, x_d]$ over $k$ in $d$ variables is the set of all polynomials formed with $d$ variables and coefficients in $k$.

**Example (Polynomial Ring)**

$3x^2y + z^3 \in \mathbb{Q}[x, y, z]$, but $\sqrt{2}x^2 \notin \mathbb{Q}[x, y, z]$.

We can multiply and add polynomials in $R$ as normal. They obey the standard commutative, associative, and distributive laws.

## Introduction

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* What are Betti Numbers?

# » What is an Ideal?

**Definition (Ideal)**

A subset $\emptyset \neq I \subset k[x_1, \ldots, x_d]$ is an ideal if it satisfies:

* if $f, g \in I$, then $f + g \in I$
* If $f \in I$ and $h \in k[x_1, \ldots, x_d]$, then $h \cdot f \in I$.

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* If $f \in I$ and $h \in k[x_1, \ldots, x_d]$, then $h \cdot f \in I$.

**Definition (Ideal from Generators)**

Let $f_1, ..., f_s$ be polynomials in $k[x_1, ..., x_d]$ and set

$$(f_1, ..., f_s) = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, ..., h_s \in k[x_1, ..., x_d] \right\}$$

Then $(f_1, ..., f_s)$ is an ideal.

**Example (Maximal Ideal)**

$(x_1, \ldots, x_d)$ is called the homogeneous maximal ideal. It consists of all polynomials with constant term 0.

## Introduction

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* What are Betti Numbers?

# » Repeatedly Solving (Resolving) Systems of Polynomials

Set $R = k[x_1, \ldots, x_d]$. Let $R^p = \{$all $p$-tuples of polynomials in $R\}$.

We wish to solve a linear system of equations over the polynomial ring $R$. The following are equivalent:

Solve the system $BX = 0$ over $R$, where $B$ is a matrix

$R^p \xrightarrow{B} R^q$ with entries in $R$

$$\Longleftrightarrow$$

Describe $\mathrm{Null}(B)$

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$$\iff$$

Describe $\mathrm{Null}(B)$

$R^p$ is similar to a vector space; null spaces may not have a basis.

# » Repeatedly Solving (Resolving) Systems of Polynomials

Null($B$) may be generated by another matrix $C$, in which case our goal is to describe Null($C$). We get a sequence

$$R^l \xrightarrow{C} R^p \xrightarrow{B} R^q$$

such that Col($C$) = Null($B$).

But then Null($C$) may be generated by another matrix $D$, so we continue to obtain a sequence

$$R^k \xrightarrow{D} R^l \xrightarrow{C} R^p \xrightarrow{B} R^q$$

so that Col($D$) = Null($C$).

This process continues, resulting in a longer sequence that we call a *free resolution*.

# » What is a Minimal Free Resolution?

**Definition (Free Resolution)**

Set $R = k[x_1, \ldots, x_d]$. A *free resolution* is a sequence

$$R^{\beta_0} \xleftarrow{d_1} R^{\beta_1} \xleftarrow{d_2} R^{\beta_2} \xleftarrow{d_3} \ldots \xleftarrow{d_p} R^{\beta_p} \ldots$$

where each $d_i$ is a matrix with (homogenous) entries in $R$,

* *(Exactness)* $\mathrm{Null}(d_i) = \mathrm{Col}(d_{i+1})$ for each $i \geq 1$,
* $\beta_i =$ the number of columns of $d_i$, and
* $\beta_{i-1} =$ the number of rows of $d_i$.

**Definition (Minimal Free Resolution)**

A free resolution is *minimal* if each of the entries of the matrices $d_i$ is in the homogeneous maximal ideal $\mathfrak{m}$ of $R$.

# » What is a Minimal Free Resolution

## Example Minimal Free Resolution

The minimal free resolution

$$R^1 \xleftarrow{d_1=\begin{bmatrix} x^3 & xy & y^5 \end{bmatrix}} R^3 \xleftarrow{d_2=\begin{bmatrix} y & 0 \\ -x^2 & -y^4 \\ 0 & x \end{bmatrix}} R^2 \xleftarrow{d_3=0} 0.$$

has

$$\beta_0 = 1, \beta_1 = 3, \beta_2 = 2.$$

Since $d_1$ has one row, we can obtain an ideal from its entries by setting $I = (x^3, xy, y^5) \subseteq R = \mathbb{Q}[x, y]$.

We say that this is the resolution of $R/I$.

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# What are Betti Numbers?

**Definition (Betti Number)**

The $i$-th *Betti number* of a minimal free resolution is $\beta_i$.

**Definition (Betti Sequences)**

The Betti sequence is $(\beta_i)_{i\geq 0}$, also written $(\beta_0, \ldots, \beta_p)$ if $\beta_i = 0$ for $i > p$.

**Definition (Ordering Betti Sequences)**

For two Betti sequences $(\beta_i)_{i\geq 0}$ and $(\beta'_i)_{i\geq 0}$, we say that $(\beta_i)_{i\geq 0} \geq (\beta'_i)_{i\geq 0}$ if $\beta_i \geq \beta'_i$ for all $i \geq 0$.

**Example Betti Sequence Order**

We have $(1, 3, 3, 1) \geq (1, 3, 2)$ since $1 \geq 1, 3 \geq 3, 3 \geq 2$, and $1 \geq 0$.

# What are Betti Numbers?

## Example Betti Table

The Betti numbers of the free resolution

$$R^1 \xleftarrow{d_1} R^3 \xleftarrow{d_2} R^2 \xleftarrow{d_3} 0.$$

are arranged together into *a Betti table*

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 1 | 3 | 2 |
| 0: | 1 | . | . |
| 1: | . | 1 | . |
| 2: | . | 1 | 1 |
| 3: | . | . | . |
| 4: | . | 1 | 1 |

The row labeled "total" gives Betti numbers $\beta_0 = 1, \beta_1 = 3, \beta_2 = 2$ with Betti sequence $(1, 3, 2)$.

### Definition (Regularity)

The *Castelnuovo-Mumford regularity* $\operatorname{reg}(R/I)$ is the maximum index of a nonzero row in the Betti table of the minimal free resolution of $R/I$.

Informally, $\operatorname{reg}(R/I)$ is a measurement of the highest degrees of the entries of the matrices appearing in the resolution.

### Example Regularity

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 1 | 3 | 2 |
| 0: | 1 | . | . |
| 1: | . | 1 | . |
| 2: | . | 1 | 1 |
| 3: | . | . | . |
| 4: | . | 1 | 1 |

The regularity is $\operatorname{reg}(R/I) = 4$.

## Betti numbers of quadratic ideals

* Computationally Generating Betti Tables
* Theoretical Approach

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## » Computations (Two variables)

1. All possible Betti tables for ideals of $\mathbb{F}_2[x, y]$ generated by homogeneous quadratics were computed.

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1. All possible Betti tables for ideals of $\mathbb{F}_2[x, y]$ generated by homogeneous quadratics were computed.
2. There are only $7$ such quadratics and thus, only $127$ nonempty generating sets.
3. These were generated using M2 and it was found that there are only $15$ distinct ideals and $4$ distinct Betti tables as below.

$$\begin{array}{lcc} & 0 & 1 \\ \text{total:} & 1 & 1 \\ \text{0:} & 1 & . \\ \text{1:} & . & 1 \end{array} \Rightarrow 7, \quad \begin{array}{lccc} & 0 & 1 & 2 \\ \text{total:} & 1 & 2 & 1 \\ \text{0:} & 1 & . & . \\ \text{1:} & . & 2 & 1 \end{array} \Rightarrow 3,$$

$$\begin{array}{lccc} & 0 & 1 & 2 \\ \text{total:} & 1 & 2 & 1 \\ \text{0:} & 1 & . & . \\ \text{1:} & . & 2 & . \\ \text{2:} & . & . & 1 \end{array} \Rightarrow 3, \quad \begin{array}{lccc} & 0 & 1 & 2 \\ \text{total:} & 1 & 3 & 2 \\ \text{0:} & 1 & . & . \\ \text{1:} & . & 3 & 2 \end{array} \Rightarrow 1$$

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1. Trying the naïve calculations as earlier doesn't work since there are $63$ quadratics giving us $2^{63} - 1 > 9 \cdot 10^{18}$ possible generating sets.

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3. This brings down the number to simply $2824$ ideals.
4. This was computable in M2 and we get a total of $15$ distinct Betti tables, as depicted in the next slides.

# » Betti tables (three variables)

$$
\begin{array}{rcc} & 0 & 1 \\ \text{total:} & 1 & 1 \\ \text{0:} & 1 & . \\ \text{1:} & . & 1 \end{array} \Rightarrow 63, \quad
\begin{array}{rccc} & 0 & 1 & 2 \\ \text{total:} & 1 & 2 & 1 \\ \text{0:} & 1 & . & . \\ \text{1:} & . & 2 & 1 \end{array} \Rightarrow 49, \quad
\begin{array}{rccc} & 0 & 1 & 2 \\ \text{total:} & 1 & 2 & 1 \\ \text{0:} & 1 & . & . \\ \text{1:} & . & 2 & . \\ \text{2:} & . & . & 1 \end{array} \Rightarrow 602,
$$

$$
\begin{array}{rcccc} & 0 & 1 & 2 & 3 \\ \text{total:} & 1 & 3 & 3 & 1 \\ \text{0:} & 1 & . & . & . \\ \text{1:} & . & 3 & 1 & . \\ \text{2:} & . & . & 2 & 1 \end{array} \Rightarrow 364, \quad
\begin{array}{rccc} & 0 & 1 & 2 \\ \text{total:} & 1 & 3 & 2 \\ \text{0:} & 1 & . & . \\ \text{1:} & . & 3 & 2 \end{array} \Rightarrow 176, \quad
\begin{array}{rcccc} & 0 & 1 & 2 & 3 \\ \text{total:} & 1 & 3 & 3 & 1 \\ \text{0:} & 1 & . & . & . \\ \text{1:} & . & 3 & 3 & 1 \end{array} \Rightarrow 7,
$$

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 3 | 3 | 1 |
| 0: | 1 | . | . | . |
| 1: | . | 3 | . | . |
| 2: | . | . | 3 | . |
| 3: | . | . | . | 1 |

$\Rightarrow 512$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 3 | 4 | 2 |
| 0: | 1 | . | . | . |
| 1: | . | 3 | . | . |
| 2: | . | . | 4 | 2 |

$\Rightarrow 336$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 4 | 5 | 2 |
| 0: | 1 | . | . | . |
| 1: | . | 4 | 2 | . |
| 2: | . | . | 3 | 2 |

$\Rightarrow 448$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 4 | 4 | 1 |
| 0: | 1 | . | . | . |
| 1: | . | 4 | 3 | . |
| 2: | . | . | 1 | 1 |

$\Rightarrow 154$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 4 | 4 | 1 |
| 0: | 1 | . | . | . |
| 1: | . | 4 | 4 | 1 |

$\Rightarrow 49$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 5 | 6 | 2 |
| 0: | 1 | . | . | . |
| 1: | . | 5 | 5 | 1 |
| 2: | . | . | 1 | 1 |

$\Rightarrow 28$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 5 | 5 | 1 |
| 0: | 1 | . | . | . |
| 1: | . | 5 | 5 | . |
| 2: | . | . | . | 1 |

$\Rightarrow 28$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 5 | 6 | 2 |
| 0: | 1 | . | . | . |
| 1: | . | 5 | 6 | 2 |

$\Rightarrow 7$,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 6 | 8 | 3 |
| 0: | 1 | . | . | . |
| 1: | . | 6 | 8 | 3 |

$\Rightarrow 1$

One may refine the above collection and only look at Betti tables corresponding to a fixed codimension.

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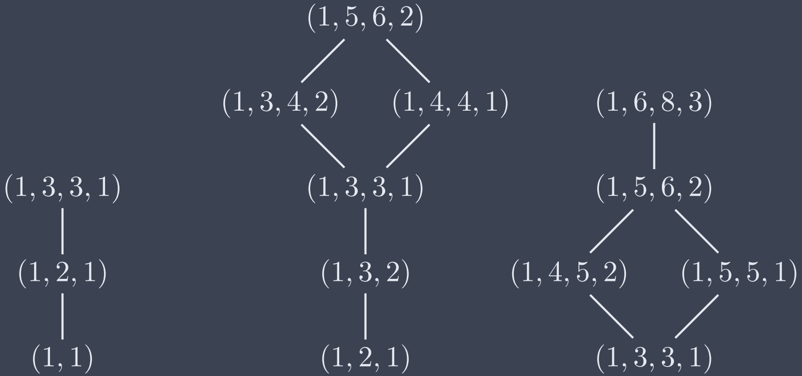


**(a)** Codimension 1 **(b)** Codimension 2 **(c)** Codimension 3

**Figure:** Hasse diagrams for Betti sequences in 3 variables

## Betti numbers of quadratic ideals

* Computationally Generating Betti Tables
* Theoretical Approach

# Koszul Complex

**Definition**

Let $R$ be a ring. We say that $a$ is a *zero divisor* in $R$ if $ab = 0$ for some nonzero $b \in R$. Otherwise $a$ is a *non-zero divisor* in $R$.

**Proposition**

*Let $R = K[x_1, \ldots, x_d]$, and $f_1, \ldots, f_n \in (x_1, \ldots, x_d)$ such that $f_1$ is a non-zero divisor in $R$, and $f_i$ is a non-zero divisor in $R/(f_1, \ldots, f_{i-1})$ for all $i \in \{2, \ldots, n\}$. Then the following gives a minimal free resolution of $R/(f_1, \ldots, f_n)$:*

$$0 \to R \xrightarrow{a_n} R^{\binom{n}{n-1}} \xrightarrow{a_{n-1}} R^{\binom{n}{n-2}} \to \ldots \to R^{\binom{n}{1}} \xrightarrow{a_1} R.$$

# » Mapping Cones

Let $f_1, f_2, f_3 \in \mathbb{F}_2[x, y, z]$. Then, we have the following exact sequence

$$0 \to R/((f_1, f_2) : (f_3)) \to R/(f_1, f_2) \to R/(f_1, f_2, f_3) \to 0.$$

The theory of mapping cones tells us that generally speaking, given minimal free resolutions of the first two terms in the exact sequence, we can determine the minimal free resolution of the last term. The Betti tables "add."

# » Ideals with 1 Generator

This is the simplest case. Indeed, given a nonzero $f \in (x_1, \ldots, x_t)$, we simply have

$$0 \to R \xrightarrow{d=(f)} R$$

as a minimal free resolution of $R/(f)$.

Thus, the betti table of $R/(f)$ is

| | 0 | 1 |
|---|---|---|
| total: | 1 | 1 |
| 0: | 1 | . |
| 1: | . | 1 |

# » Ideals with 2 Generators

Quadratic ideals $(f_1, f_2)$ in $\mathbb{F}_2[x, y, z]$ fall into two categories:

1. If $f_2$ is a nonzerodivisor in $R/(f_1)$, then by the Koszul complex, the Betti table corresponding to $R/I$ is

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 1 | 2 | 1 |
| 0: | 1 | . | . |
| 1: | . | 2 | . |
| 2: | . | . | 1 |

2. If $f_2$ is a zerodivisor in $R/(f_1)$, the Betti table corresponding to $R/I$ is

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 1 | 2 | 1 |
| 0: | 1 | . | . |
| 1: | . | 2 | 1 |

# » Ideals with 3 Generators

To resolve all quadratic ideals $(f_1, f_2, f_3)$ in $\mathbb{F}_2[x, y, z]$, we divide the ideals into four cases:

| | $f_2$ zero divisor in $R/(f_1)$ | $f_3$ zero divisor in $R/(f_1, f_2)$ | Number | Example |
|---|---|---|---|---|
| Case 1 | | | 512 | $(x^2, y^2, z^2)$ |
| Case 2 | ✓ | | 297 | $(x^2, xy, z^2)$ |
| Case 3 | ✓ | ✓ | 874 | $(x^2, y^2, xy)$ |
| Case 4 | | ✓ | 161 | $(x^2, xy, y^2)$ |

In all, there are 1844 ideals with three generators in $\mathbb{F}_2[x, y, z]$. Except for a special case in case $4$, we managed to use Koszul Complex and Mapping Cone to obtain the Betti Tables for all the ideals in the four cases.

## Axial constants

* Background
* Growth of Axial Constants

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# » Monomial Orders

Let $R = K[x_1, \ldots, x_d]$. For each $\alpha = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\geq 0}$ set $x^\alpha = x_1^{a_1} \ldots x_d^{a_d}$.

**Definition (Graded Reverse Lexicographic (Grevlex) Order)**

For each $x^\alpha, x^\beta \in R$, $x^\alpha > x^\beta$ provided that $\deg(x^\alpha) > \deg(x^\beta)$ or if $\deg(x^\alpha) = \deg(x^\beta)$ and the last nonzero entry of $\alpha - \beta$ is negative.

**Example**

Let $R = \mathbb{Q}[x, y, z]$.

* $x^4yz > x^2y$ because $6 > 3$
* $x^2y > xyz$ because $(2, 1, 0) - (1, 1, 1) = (1, 0, -1)$

# » Initial Ideals

**Definition (Initial Term)**

Given $f \in R \setminus \{0\}$, the initial term of $f$, denoted $\mathrm{in}(f)$, is the largest monomial appearing as a term in $f$.

**Example**

Let $R = \mathbb{Q}[x, y, z]$. Then $\mathrm{in}(x^4yz + x^2y + xyz) = x^4yz$

**Definition (Initial Ideal)**

Let $I$ be an ideal. Then the initial ideal of $I$, is defined to be

$$\mathrm{in}(I) := (\{\mathrm{in}(f) \mid f \in I\})$$

# » Initial Ideals

Initial ideals depend on the choice of coordinate system.

**Example**

Let $I = (x^2, y^2) \subseteq \mathbb{Q}[x, y]$. A change of coordinates $x \mapsto x + y$ and $y \mapsto x - y$, results in

$$I' = ((x + y)^2, (x - y)^2).$$

Taking initial ideals,

$$\mathrm{in}(I) = (x^2, y^2) \text{ and } \mathrm{in}(I') = (x^2, xy, y^3).$$

This motivates the introduction of generic initial ideals.

# » Generic Initial Ideals

**Definition (Generic Initial Ideals)**

The generic initial ideal of a homogeneous ideal $I$, denoted $\operatorname{gin}(I)$, is the ideal obtained by first applying a sufficiently general linear change of coordinates on $R$, then taking the initial ideal.

**Example**

Let $I = (x^2, y^2) \subseteq \mathbb{Q}[x, y]$. Taking a sufficiently general change of coordinates $x \mapsto x + y$ and $y \mapsto x - y$ we have

$$\operatorname{gin}(I) = \operatorname{in}((x + y)^2, (x - y)^2) = (x^2, xy, y^3).$$

# » Powers of Ideals

We study how the operation of taking the generic initial ideal interacts with the operation of multiplication.

**Definition ($n$-th Power of an Ideal)**

For $n \in \mathbb{N}$ the $n$-th power of an ideal $I$ is the ideal generated by the $n$-fold products of elements of $I$, i.e.,

$$I^n = (f_1 f_2 \ldots f_n | f_i \in I).$$

**Example**

Suppose $I = (x^2, y^2)$. It follows that $I^2 = (x^4, x^2y^2, y^4)$, $I^3 = (x^6, x^4y^2, x^2y^4, y^6)$, etc.

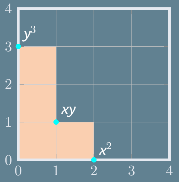
# What is an Axial Number?

**Definition**

Let $I$ be a monomial ideal and $1 \leq i \leq d$. Then we define the *i*-th *axial* number of $I$, denoted $n_i(I)$, to be equal to the smallest power of $x_i$ contained in $I$.

**Example**

Let $I = (x^2, xy, y^3)$. Then $n_1(I) = 2$ and $n_2(I) = 3$.




gin($I$)

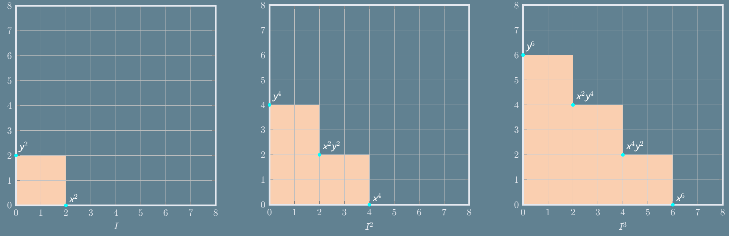
# » Axial Numbers of Powers of Ideals

**Proposition**

*Let $I$ be a monomial ideal and $1 \leq i \leq d$. Then $n_i(I^n)$ is linear.*

**Example**

Let $I = (x^2, y^2)$. Then $n_i(I^n) = 2n$.



# » What is an Axial Constant?

**Definition (Axial Constants)**

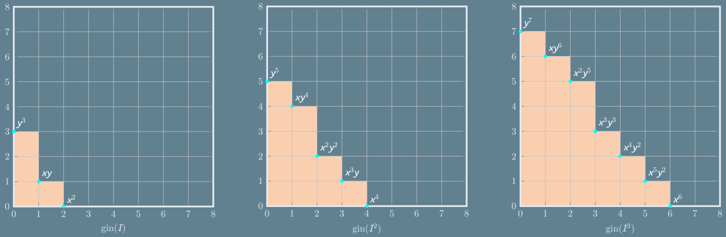
For a homogeneous ideal $I$, let $a_i(I) = n_i(\mathrm{gin}(I))$.

**Definition**

Let $I$ be a homogenous ideal. Let $a_i(n) = a_i(I^n)$.

**Example**

Let $I = (x^2, y^2) \subseteq \mathbb{Q}[x, y]$. So $\mathrm{gin}(I) = (x^2, xy, y^3)$, and we can show that $a_i(n) = 2n + i - 1$.



# » Significance

**Proposition**

*If $r$ is the largest integer for which $a_r(I)$ is defined, then $a_r(I) = \operatorname{reg}(I)$.*

**Fact**

$\operatorname{reg}(I^n)$ *is linear for* $n \gg 0$.

**Corollary**

$a_r(n)$ *is linear for* $n \gg 0$.

## Axial constants

* Background
* Growth of Axial Constants

# » Linearity of Axial Constants

Let $I$ be a homogeneous ideal.

**Theorem**

*For some $\overline{I}$ depending on $I$, $a_i(n) = \mathrm{reg}(\overline{I}^n)$.*

**Corollary (Linearity of Axial Constants)**

*$a_i(n)$ is linear for $n \gg 0$.*

**Example (First Axial Constant)**

$a_1(n) = na_1(I)$

**Example (Last Axial Constant)**

If $I = (x_1^{a_1}, \ldots, x_d^{a_d})$ and $a = \max\{a_i\}$ then

$$a_d(n) = a(n-1) + (a_1 + \ldots + a_d) - (d-1).$$

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