Numerical Analysis of the 1D Semilinear Heat Equation

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Overview

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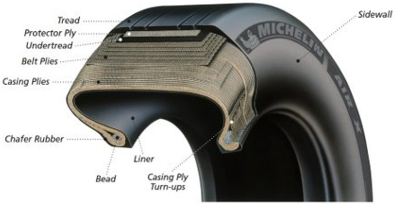

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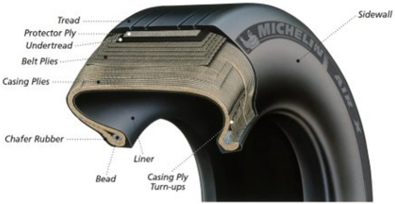

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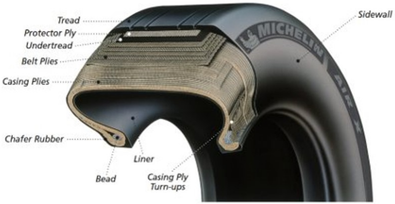

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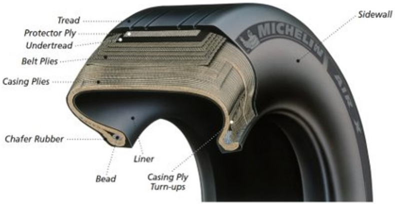

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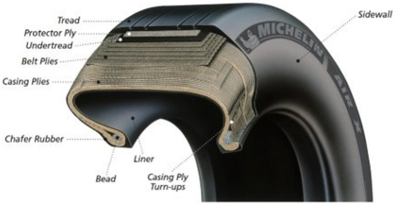

1. My four polymer-based tires are subject to mechanical vibrations.
2. In particular, the nylon cords are in an adiabatic environment.
3. The mechanical vibrations plus the adiabatic environment lead to the self-heating phenomenon.
4. Over time, this can cause permanent deformation of the material.

More Applications

The problem of self-heating arises across multiple fields and industries.

Introduction to the Semilinear Heat Equation

We consider the semilinear heat equation which is given in 1D as

$$\begin{aligned} u_t - \Delta u + \beta\,(u - u_g) &= \alpha|u|^\gamma \quad \text{on} \quad (a,b), \\ u(a,t) = u(b,t) &= u_g, \\ u(x,0) &= u_g, \end{aligned}$$

where

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Note: If $\beta = 0$, this equation is called the Fujita equation.

Modeling the Self-Heating Phenomenon

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- With certain parameters, $u(x, t)$ can grow exponentially without any bound as t increases.
- This exponential growth, or "blow-up", corresponds to material failure in the context of our problem.
- Remark: u_g is constant for simplicity, but it is okay if u_g changes in time.

Our Numerical Scheme

We rewrite the semilinear heat equation in terms of a new variable:

$$\hat{u} = u - u_g$$

We consider a backward Euler temporal discretization together with a first order explicit approximation of the nonlinear term:

$$\frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} - \Delta \hat{u}^{n+1} + \beta \hat{u}^{n+1} = \alpha \left(\hat{u} + u_g\right)^{\gamma},$$

$$\hat{u}^0 = 0,$$

where $\Delta t > 0$ denotes the time step size and $\hat{u}$ is zero on the boundary.

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- Numerical method failure directly relates to the instability of the 1D semilinear heat equation.
- Through our numerical scheme, we were able to determine sufficient conditions that give stability for specific gamma cases:
  - $0 < \gamma < 1$
  - $\gamma = 1$
  - $1 < \gamma < 3$
  - $\gamma = 3$
  - $\gamma > 3$

Theorem 1

Suppose that $\gamma = 1$ *in our algorithm and* $\beta > \frac{3\alpha}{2}$. *Then the solution is long time stable: for any given* $n \in \mathrm{N}$,

$$||\hat{u}^n|| \leq \frac{\frac{\alpha}{2}||u_g||^2}{C_p^{-2}+\beta-\frac{3\alpha}{2}},$$

where C_p^{-2} *is a Poincare constant.*

Note: For our proof, (x, y) is defined as the $L^2(\Omega)$ inner product.

Proof Outline of Theorem 1

- We multiply our algorithm by $\hat{u}^{n+1}$ and take the integral over Ω. Rewriting our first term using the polarization identity and our second term using Green's theorem, we now have that

$$\frac{1}{2\Delta t}\left(||\hat{u}^{n+1}||^2 - ||\hat{u}^n||^2 + ||\hat{u}^{n+1} - \hat{u}^n||^2\right) + ||\nabla \hat{u}^{n+1}||^2 + \beta||\hat{u}^{n+1}||^2 = \alpha\left(\left(\hat{u}^n + u_g\right), \hat{u}^{n+1}\right).$$

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- By the Cauchy-Schwarz and Young inequalities, we can upper bound the right-hand side of our equation. Dropping the nonnegative term, $\frac{1}{2\Delta t}||\hat{u}^{n+1} - \hat{u}^n||^2$ and using Poincare's inequality, we lower bound the left-hand side of our equation. We now have that

$$\left(\frac{1}{2\Delta t} + C_p^{-2} + \beta - \alpha\right)||\hat{u}^{n+1}||^2 \leq \left(\frac{\alpha}{2} + \frac{1}{2\Delta t}\right)||\hat{u}^n||^2 + \frac{\alpha}{2}||u_g||^2.$$

Proof Outline of Theorem 1 Cont'd

- Dividing both sides by $\frac{\alpha}{2} + \frac{1}{2\Delta t}$, we find that

$$\left(\frac{\frac{1}{2\Delta t} + C_p^{-2} + \beta - \alpha}{\frac{\alpha}{2} + \frac{1}{2\Delta t}} \right) ||\hat{u}^{n+1}||^2 \leq ||\hat{u}^n||^2 + \left(\frac{1}{\frac{\alpha}{2} + \frac{1}{2\Delta t}} \right) \frac{\alpha}{2} ||u_g||^2.$$

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$$\left(\frac{\frac{1}{2\Delta t}+C_p^{-2}+\beta-\alpha}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)||\hat{u}^{n+1}||^2 \leq ||\hat{u}^n||^2 + \left(\frac{1}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)\frac{\alpha}{2}||u_g||^2.$$

- Let $r := \frac{\frac{1}{2\Delta t}+C_p^{-2}+\beta-\alpha}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}$ and note that since $\beta > \frac{3\alpha}{2}$, we can now use Lemma 2.5 from Larios, et. al. to get

$$||\hat{u}^{n+1}||^2 \leq ||\hat{u}^0||^2 \left(\frac{1}{r}\right)^{n+1} + \frac{\left(\frac{1}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)\frac{\alpha}{2}||u_g||^2}{r-1} = \frac{\frac{\alpha}{2}||u_g||^2}{C_p^{-2}+\beta-\frac{3\alpha}{2}},$$

which completes the proof.

Theorem 3

Suppose that $\gamma = 3$ *in the numerical scheme. Assume that* $\Delta t \leq \min\{1, \frac{1}{\beta}\}$*, and*

$$\alpha \leq \frac{1}{2\left(|\Omega|2^\gamma\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)\frac{2}{\beta} + |\Omega|2^\gamma\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)^2\frac{4}{\beta^2}\right)}$$

Then the solution is long time stable: for any given $n \in \mathrm{N}$,

$$||\hat{u}^n|| \leq \sqrt{\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)}.$$

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  - one-dimensional Gagliardo–Nirenberg–Sobolev inequalities
    $$||f||_{L^p} \leq C_{GN}(p)||f'||_{L^2}^{\theta}||f||_{L^2}^{1-\theta} \quad \text{if} \quad p \in (2, \infty), \text{ with } \theta = \frac{p-2}{2p}$$
  - Agmon's inequality
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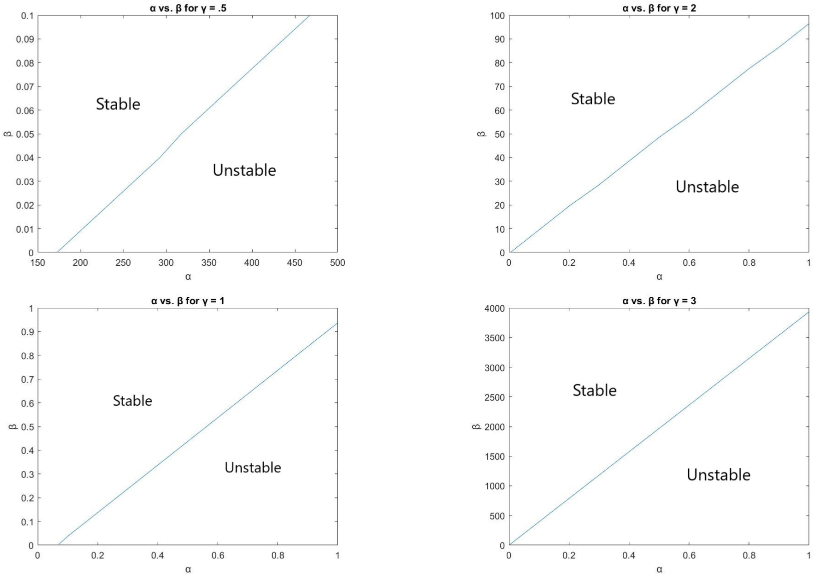
    $$||u||_{L^\infty(\Omega)} \leq C||u||_{L^2}^{\frac{1}{2}}||u||_{H^1(\Omega)}^{\frac{1}{2}}$$
- The tools utilized in this theorem provided a basis for the remaining cases (excluding $0 < \gamma < 1$).

Sufficient Stability Conditions

Gamma	Bound (if restricitions are met)
$0 < \gamma < 1$	$\|\|\hat{u}^n\|\| \leq \frac{\alpha\|\|u_g\|\|^2}{C_p^{-2}+\beta-2\alpha(\|\Omega\|+1)}$
$\gamma = 1$	$\|\|\hat{u}^n\|\| \leq \frac{\frac{\alpha}{2}\|\|u_g\|\|^2}{C_p^{-2}+\beta-\frac{3\alpha}{2}}$
$1 < \gamma < 3$	$\|\|\hat{u}^n\|\| \leq \sqrt{\frac{4}{\beta}\left(2\int_\Omega\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}\|\|u_g^\gamma\|\|^2\right)}$
$\gamma = 3$	$\|\|\hat{u}^n\|\| \leq \sqrt{\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2\|\|u_g^\gamma\|\|^2\right)}$
$\gamma > 3$	$\|\|\hat{u}^n\|\| \leq \sqrt{\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2\|\|u_g^\gamma\|\|^2\right)}$

Numerical Results



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Future Work

- Extend this work into higher dimensions.

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- The proofs are more involved for the analysis of $\gamma > 1$.
- Our numerical results align and illustrate our theorems.

Future Work

- Extend this work into higher dimensions.
- Create, analyze, and test a 1D data assimilation model for the semilinear heat equation with a penalization term.

Thank You!

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