Frame Theory and its Applications

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The Beginnings of Frame Theory

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  - Speech Recognition
  - Signal and Image Processing
  - Quantum Information Theory
  - Sampling Theory
  - Biomedical Engineering

Tools to Understand Frame Theory

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- Functional Analysis
- Operator Theory
- Numerical Linear Algebra
- Banach Space Theory
- Algebraic Geometry
- Graph Theory
- Number Theory

Definition from Finite Frame Theory

Parseval's Identity: Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for Hilbert space $\mathcal{H}^N$. Then, for all $x \in \mathcal{H}^N$

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Frame Definition

A family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in $\mathcal{H}^N$ is a *frame* if there exists $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}^N$,

$$A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2,$$

where A and B are the lower and upper frame bounds, respectively.

- If $A = B = 1$ it is a *Parseval frame.*

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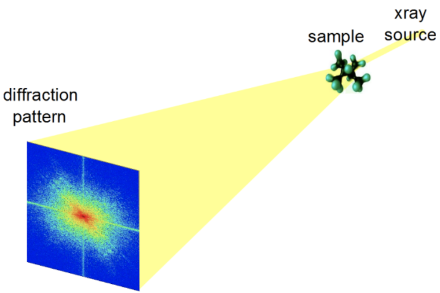
Example: Imagine you are texting your best friend...

The Phase Retrieval Problem



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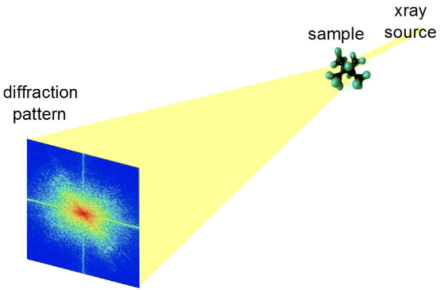

- Phase information can be lost when signals pass through a linear system.
- **Phase Retrieval:** Process of retrieving the phase of a signal from the absolute value of linear measurements known as intensity measurements.

Frame-theoretic Approach

Phase Retrieval

We say $\Phi = \{\varphi_i\}_{i=1}^M$ is said to yield phase retrieval if given $x, y \in \mathbb{R}^N$, such that

$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \text{ for } i \in \{1, \dots, M\}$$

then $x = cy$, where $c = \pm 1$.

- We can recover (or reconstruct) vectors up to a phase factor
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Definition: A set of vectors $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ has the complement property if for every partition $S \subseteq \{1, \ldots, M\}$,

$$\text{either } \operatorname{span}\{\varphi_i\}_{i \in S} = \mathbb{R}^N \text{ or } \operatorname{span}\{\varphi_i\}_{i \in S^c} = \mathbb{R}^N$$

.

Classification of Phase Retrieval

Theorem (Balan, Casazza, Edidin)

A frame $\Phi = \{\varphi_i\}_{i=1}^{M}$ in $\mathbb{R}^N$ does phase retrieval if and only if it satisfies the complement property.

- It follows that the minimal number of vectors to do phase retrieval in $\mathbb{R}^N$ is 2N-1.

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Why not $2N-2$ vectors?

For example, take $\Phi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ in $\mathbb{R}^3$. Then, we can partition Φ such that $S = \{\varphi_1, \varphi_2\}$ and $S^c = \{\varphi_3, \varphi_4\}$, neither of which span $\mathbb{R}^3$, which fails the complement property.

An Overview of *Weak* Phase Retrieval

- We only consider vectors with non-zero coordinates in the same position.

$$\begin{bmatrix} 0 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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Weak Phase Retrieval (Botelho-Andrade, Casazza, Ghoreishi, Jose, Tremain)

If for any $x = (a_1, a_2, \cdots, a_N)$ and $y = (b_1, b_2, \cdots, b_N)$ in $\mathcal{H}^N$, the equality

$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \text{ for } i \in \{1, \ldots, M\}$$

implies that for a_i and b_i, there exists a $|\theta| = 1$ such that $phase(a_i) = \theta phase(b_i)$, then the family of vectors $\{\varphi_i\}_{i=1}^M$ in $\mathcal{H}^N$ does *weak phase retrieval.*

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- Especially if we start with a lot of data!

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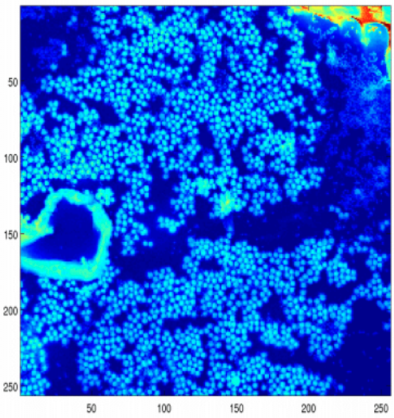
- We preform *algorithmic* phase retrieval
- Existing phase retrieval algorithms include:
  - Gerchberg-Saxton and Fienup Algorithms
  - Phaseless Recontstruciton using Frames
  - PhaseLift Methodology



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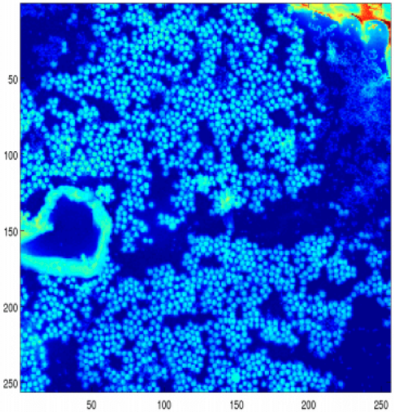

Can these algorithms also work for weak phase retrieval?

Existing Phase Retrieval Algorithms

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  - Alternating Projection Method
  - Utilized the properties of Fourier transforms
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  - We can reconstruct vectors using the magnitude of frame coefficients without given phase information
- PhaseLift Methodology
  - Instead of recovering a signal (vector), we *lift* the problem to retrieving a low-rank matrix
  - Requires convex optimization and semidefinite programming

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