Frame Theory and its Applications

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Later, in the 1980’s, frame theory was revived by Daubechies, Grossman, and Meyer to investigate problems in signal processing. Since then, frames have become an active area of research, theoretically, and in fields including:

- Speech Recognition
- Signal and Image Processing
- Quantum Information Theory
- Sampling Theory
- Biomedical Engineering
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Tools to Understand Frame Theory

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- Functional Analysis
- Operator Theory
- Numerical Linear Algebra
- Banach Space Theory
- Algebraic Geometry
- Graph Theory
- Number Theory
Definition from Finite Frame Theory

**Parseval's Identity:** Let \( \{e_i\}_{i=1}^N \) be an orthonormal basis for Hilbert space \( \mathcal{H}^N \). Then, for all \( x \in \mathcal{H}^N \)

\[
\|x\|^2 = \sum_{i=1}^{N} |\langle x, e_i \rangle|^2.
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Frame Definition

A family of vectors \( \Phi = \{\varphi_i\}_{i=1}^M \) in \( \mathcal{H}^N \) is a frame if there exists \( 0 < A \leq B < \infty \) such that, for every \( x \in \mathcal{H}^N \),

\[
A\|x\|^2 \leq \sum_{i=1}^M |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2,
\]

where \( A \) and \( B \) are the lower and upper frame bounds, respectively.

- If \( A = B = 1 \) it is a Parseval frame.
Question: Why do we study frames?

- Frames gives us a way to reconstruct a vector
- This reconstruction method is stable, linear, and continuous
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☐ Frames gives us a way to reconstruct a vector
☐ This reconstruction method is stable, linear, and continuous
☐ Frames maintain characteristics of orthonormal bases, but are more flexible in design
☐ Frames allow for redundancy, making them resilient to noise or information loss
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Example: Imagine you are texting your best friend...
Phase information can be lost when signals pass through a linear system.
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- **Phase Retrieval**: Process of retrieving the phase of a signal from the absolute value of linear measurements known as intensity measurements.
We say \( \Phi = \{\varphi_i\}_{i=1}^M \) is said to yield phase retrieval if given \( x, y \in \mathbb{R}^N \), such that
\[
|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \quad \text{for} \quad i \in \{1, \ldots, M\}
\]
then \( x = cy \), where \( c = \pm 1 \).

- We can recover (or reconstruct) vectors up to a phase factor
- Checking if vectors do phase retrieval can be computationally difficult, so we instead use the complement property
Frame-theoretic Approach

Phase Retrieval

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$$|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \text{ for } i \in \{1, \ldots, M\}$$

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**Definition:** A set of vectors $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ has the complement property if for every partition $S \subseteq \{1, \ldots, M\}$,

$$\text{either } \text{span}\{\varphi_i\}_{i \in S} = \mathbb{R}^N \text{ or } \text{span}\{\varphi_i\}_{i \in S^c} = \mathbb{R}^N$$
Classification of Phase Retrieval

**Theorem (Balan, Casazza, Edidin)**

A frame $\Phi = \{\varphi_i\}_{i=1}^M$ in $\mathbb{R}^N$ does phase retrieval if and only if it satisfies the complement property.

- It follows that the minimal number of vectors to do phase retrieval in $\mathbb{R}^N$ is $2N-1$. 
Theorem (Balan, Casazza, Edidin)

A frame $\Phi = \{\varphi_i\}_{i=1}^M$ in $\mathbb{R}^N$ does phase retrieval if and only if it satisfies the complement property.

It follows that the minimal number of vectors to do phase retrieval in $\mathbb{R}^N$ is $2N-1$.

Why not $2N - 2$ vectors?

For example, take $\Phi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ in $\mathbb{R}^3$. Then, we can partition $\Phi$ such that $S = \{\varphi_1, \varphi_2\}$ and $S^c = \{\varphi_3, \varphi_4\}$, neither of which span $\mathbb{R}^3$, which fails the complement property.
An Overview of *Weak* Phase Retrieval

- We only consider vectors with non-zero coordinates in the same position.

\[
\begin{bmatrix}
0 \\
a_2
\end{bmatrix}, \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]
An Overview of Weak Phase Retrieval

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- We need to check the signs of the entries \((a_i = \pm b_i)\). For example,

\[
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} -b_1 \\ -b_2 \end{bmatrix}
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Weak Phase Retrieval (Botelho-Andrade, Casazza, Ghoreishi, Jose, Tremain)

If for any \(x = (a_1, a_2, \cdots, a_N)\) and \(y = (b_1, b_2, \cdots, b_N)\) in \(\mathcal{H}^N\), the equality

\[
|\langle x, \varphi_i \rangle| = |\langle y, \varphi_i \rangle| \quad \text{for} \ i \in \{1, \ldots, M\}
\]

implies that for \(a_i\) and \(b_i\), there exists a \(|\theta| = 1\) such that \(\text{phase}(a_i) = \theta \text{phase}(b_i)\), then the family of vectors \(\{\varphi_i\}_{i=1}^M\) in \(\mathcal{H}^N\) does weak phase retrieval.
Phase retrieval calculations can be computationally taxing by hand especially if we start with a lot of data!
Phase retrieval calculations can be computationally taxing by hand. Especially if we start with a lot of data! We perform *algorithmic* phase retrieval. Existing phase retrieval algorithms include:

- Gerchberg-Saxton and Fienup Algorithms
- Phaseless Reconstruction using Frames
- PhaseLift Methodology
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Can these algorithms also work for weak phase retrieval?
Existing Phase Retrieval Algorithms

- Algorithmic phase retrieval was pioneered by Gerchberg and Saxton, and later, Fienup during the 1970’s
  - Alternating Projection Method
  - Utilized the properties of Fourier transforms
  - Computationally challenging

- More recently, a frame-theoretic approach to algorithmic phase retrieval was established
  - We can reconstruct vectors using the magnitude of frame coefficients without given phase information
  - PhaseLift Methodology
    - Instead of recovering a signal (vector), we lift the problem to retrieving a low-rank matrix
    - Requires convex optimization and semidefinite programming
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References


