The $p$-adic Numbers

Uyen Tran

(University of Nebraska - Lincoln)
Why?

**Polynomials Over \( \mathbb{C} \)**

- Rational functions
  \[ f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \]
- Any polynomial over \( \mathbb{C} \) can be written of a unique product of the form:
  \[ P(x) = c(x - \alpha_1) \cdots (x - \alpha_m) \]
- We can consider \( (x - \alpha) \) as a "prime" in the set of polynomials over \( \mathbb{C} \).
- **Question:** We can rewrite any polynomials as a power series around \((x - \alpha)\). Can we do something like that with integers and prime numbers?

\[ P(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i \geq 0} b_i (x - \alpha)^i \]

**Integers**

- Rational numbers
  \[ q = \frac{a}{b}, \quad a, b \in \mathbb{Z}, b \neq 0 \]
- Any integer can be written as a product of prime numbers:
  \[ a = p_1 p_2 \cdots p_n \]
- **Answer:** Yes!
  
  \[ 321 = 3 \times 10^2 + 2 \times 10 + 1 \]
  
  \[ 321 = 6 \times 7^2 + 3 \times 7 + 6 \]
  
  \( p \)-adic number!
What are p-adic numbers?

**Defn:** Fix a prime number $p$. A **p-adic number** is a formal sum of the form:

$$a_n p^n_0 + a_{n+1} p^{n+1} + \ldots$$

where $a_i \in \{0, 1, \ldots, p - 1\}$ for all $i$.

Examples:

- $p = 3$, $1 + 2 \times 3$
- $p = 5$, $2 \times 5^{-1} + 1 \times 5$
- $p = 5$, $4 + 4 \times 5 + 4 \times 5^2 + 4 \times 5^3 + \ldots$

**Notes:**

- Formal sum: Don’t try to put it into a calculator!
- $n_0$ can be zero, or a positive integer, or a negative integer.
- A $p$-adic number could be a finite or infinite expression.
Addition and Multiplication

Fix a prime number $p$. Given two $p$-adic numbers $x = \sum_{i \geq n_0} a_i p^i$, $y = \sum_{j \geq m_0} b_j p^j$.

- Addition:
  \[ x + y = \sum_{i \geq n_0} (a_i + b_i) p^i \]

- Multiplication:
  \[ xy = \sum_{k \geq n_0 + m_0} \left( \sum_{i+j=k} a_i b_j \right) p^k \]

- Example: Fix $p = 5$, $x = 1 + 3 \times 5$, $y = 2 + 1 \times 5$
  \[ x + y = (1 + 2) + (3 + 1) \times 5 = 3 + 4 \times 5 \]
**p-adic absolute value**

**Defn:** Fix a prime number \( p \). Given a \( p \)-adic number \( x = a_np^n + a_{n+1}p^{n+1} + \cdots \),

The \textbf{p-adic absolute value of} \( x \), denoted \(|x|_p\), is

\[
|x|_p = \frac{1}{p^n}
\]

where \( p^n \) is the first power of \( p \) that has a non-zero coefficient.

**Examples:**

- \( p = 5, x = 2p + 4p^2 + 4p^6 \), then \(|x|_5 = \frac{1}{p} \).
- \( p = 5, x = 1 + 3p + 3p^2 \), then \(|x|_5 = \frac{1}{p^0} \).
**p-derivations**

Fix a prime number $p$.

Let $R$ be a ring where $p = 1 + \cdots + 1$ is not a zero-divisor.

A function $\delta: R \to R$ is a **p-derivation** if

- $\delta(1) = 0$
- $\delta(ab) = \delta(a)b^p + a^p \delta(b) + p\delta(a)\delta(b)$
- $\delta(a + b) = \delta(a) + \delta(b) + \frac{a^p + b^p - (a+b)^p}{p}$

**Derivatives of polynomials**

Recall that derivatives of polynomials also have a similar set of rules:

- $\frac{d}{dx}(1) = 0$
- $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(g(x))$
- $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$
**p-derivations**

- Following the rules, we find the unique p-derivation over the integers is the function delta $\delta: \mathbb{Z} \to \mathbb{Z}$ given by
  \[ \delta(n) = \frac{n - n^p}{p} \]
- If the highest power of $p$ that divides $n$ is greater than 0, taking the p-derivation of $n$ will decrease that higher power of $p$ that divides $n$ by 1.
- Example:
  Fix $p = 2$, $\delta(6) = \frac{6 - 36}{2} = -15$

**Derivatives of polynomials**

- If the degree of a polynomial is not zero, taking the derivative of that polynomial will decrease the degree by one.
Thank You :)