Bounds on the Fractional Chromatic Number of a Graph

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Introduction

My research involved looking into fractional coloring.

We found a bound on the fractional chromatic number of a graph using its eigenvalues:

$$\chi_f(G) \geq \frac{\lambda_1 - \lambda_n}{1 - t(1 + \lambda_n)}$$

It is based off of Hoffman's Lower Bound for the chromatic number.

Graph Coloring

Recall regular graph coloring. We color vertices so adjacent vertices have different colors.
For a graph G, then the chromatic number $\chi(G)$ is the smallest number of colors that can make a proper coloring of G.

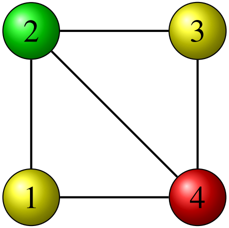


Figure: Diamond

Graph Coloring

Consider C_6 and C_5. We see that

$$\chi(C_6) = 2, \quad \chi(C_5) = 3$$

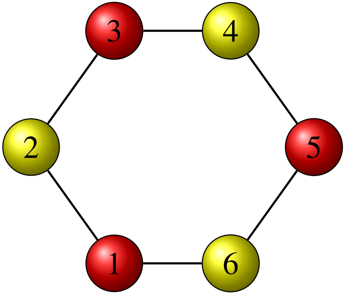
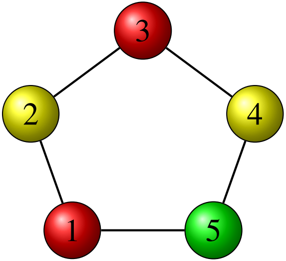


Figure: Even Cycle C_6



Odd Cycle C_5

Fractional Graph Coloring

Fractional graph coloring assigns b colors to each vertex, and adjacent vertices must have disjoint sets of colors.
The b-fold chromatic number $\chi_b(G)$ is the smallest number of colors that can make a proper coloring of G.

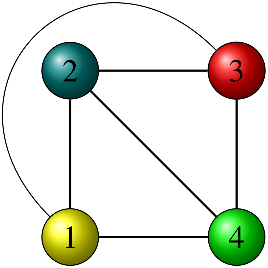


Figure: Complete Graph K_4

Fractional Graph Coloring

Consider C_6 and C_5. And let $b = 2$. Then

$$\chi_b(C_6) = 4, \quad \chi_b(C_5) = 5$$

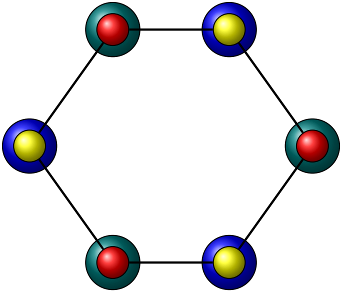
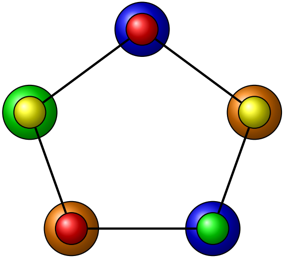


Figure: Even Cycle C_6



Odd Cycle C_5

Fractional Graph Coloring

The fractional chromatic number $\chi_f(G)$ is defined as

$$\chi_f(G) = \lim_{b \to \infty} \frac{\chi_b(G)}{b}$$

There is a finite b such that $\chi_f(G) = \frac{\chi_b(G)}{b}$.

Examples

Consider C_6 and C_5. And let $b = 2$. Recall

$$\chi_b(C_6) = 4, \chi_b(C_5) = 5$$

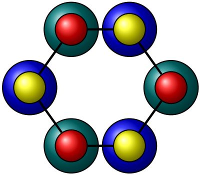
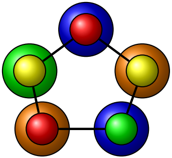


Figure: Even Cycle C_6



Odd Cycle C_5

We divide $\chi_b(C_6)$ and $\chi_b(C_5)$ by $b = 2$, and see that

$$\chi_f(C_6) = 2, \chi_f(C_5) = \frac{5}{2}$$

Strong Product

We found that coloring the strong product of G and K_b is analogous to using fractional coloring with G. Let $b = 3$.

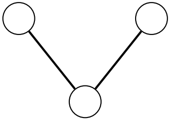
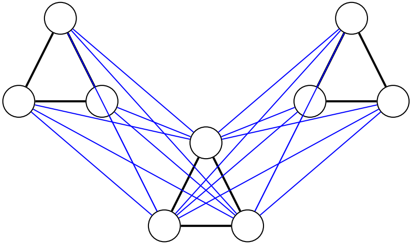


Figure: G



$G \otimes K_3$

This allows us to apply more coloring results to fractional coloring.

Hoffman's Lower Bound

Hoffman's Lower Bound (HLB) gives a lower bound for the value of $\chi(G)$ using the eigenvalues of G.

The eigenvalues of G are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

$$\chi(G) \geq 1 + \frac{\lambda_1}{|\lambda_n|}$$

Thanks to the strong product, we can apply this to fractional coloring!

Result

We found a fractional version of Hoffman's Lower Bound, which gives a lower bound for the value of $\chi_f(G)$ using the eigenvalues of G.

$$\chi_f(G) \geq \frac{\lambda_1 - \lambda_n}{1 - t(1 + \lambda_n)} \geq \chi_t(G)(1 + \lambda_n) + \lambda_1 - \lambda_n$$

Result

Recall HLB: $\chi(G) \geq 1 + \frac{\lambda_1}{|\lambda_n|}$

We solve for the eigenvalues of $G \otimes K_b$.
Then, using HLB and the strong product, we find

$$\chi_b(G) = \chi(G_b) \geq 1 + \frac{b\lambda_1 + b - 1}{|b\lambda_n + b - 1|} = \frac{b\lambda_1 - b\lambda_n}{1 - b(\lambda_n + 1)}.$$

So,

$$\chi_b(G) \geq \frac{b\lambda_1 - b\lambda_n}{1 - b(\lambda_n + 1)}.$$

Result

Recall that there is always a finite b where $\chi_f(G) = \frac{\chi_b(G)}{b}$. We will represent this ideal value for b as b^*.
So,

$$\chi_{b^*}(G) \geq \frac{b^*\lambda_1 - b^*\lambda_n}{1 - b^*(\lambda_n + 1)}.$$

Now we plug this into $\chi_f(G) = \frac{\chi_b(G)}{b}$ to find a bound on $\chi_f(G)$. (Divide both sides by b^*.) We get

$$\chi_f(G) = \frac{\chi_{b^*}(G)}{b^*} \geq \frac{\lambda_1 - \lambda_n}{1 - b^*(\lambda_n + 1)}.$$

Examples

Consider C_6 and C_5.

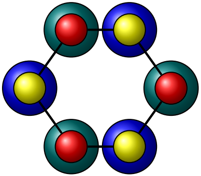
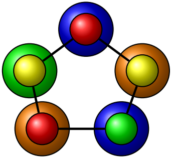


Figure: Even Cycle C_6



Odd Cycle C_5

Recall,

$$\chi_f(C_6) = 2, \chi_f(C_5) = \frac{5}{2}$$

Examples

Recall,

$$\chi_f(C_6) = 2, \chi_f(C_5) = \frac{5}{2}$$

The largest and smallest eigenvalues of C_6 and C_5 are, respectively,

$$2, \ -2, \text{ and } 2, \frac{-\sqrt{5}-1}{2}$$

When we use our bound, we get that, respectively,

$$\chi_f(C_6) \geq \frac{2-(-2)}{1-((-2)+1)} = 2$$

$$\chi_f(C_5) \geq \frac{2-(\frac{-\sqrt{5}-1}{2})}{1-2((\frac{-\sqrt{5}-1}{2})+1)} \approx 1.618$$

$$\chi_f(C_6) \geq 2 \text{ and } \chi_f(C_5) \geq 1.618$$

We see that the actual values fit these inequalities.

Future

How can we tell what b^* is?

How helpful is this bound?

What other lower bounds can we find?

What else can we do with the connection to the strong product?

Conclusion

Thanks to my mentor Sudipta Mallik for working on this research with me.

Thanks to Northern Arizona University for research funding.

Thanks to Jeff Rushall for supporting me in this process.

And thank you for listening!