{Research News

COMMUTATIVE ALGEBRA

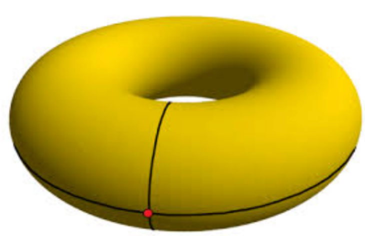
Walker resolves Betti number conjecture

Mathematicians have been fascinated with polynomials for centuries. Surely you have fond memories of those critters, expressions like $f(x) = x^2 - 1$ or $h(x, y) = 5x^2 + 3y^2 - 2xy$ that you spent countless hours graphing, factoring, differentiating and integrating back in calculus. It turns out, there are many questions about polynomials we still don't know the answer to. But thanks to the efforts of Professor Mark Walker, that list of open questions is now slightly shorter.

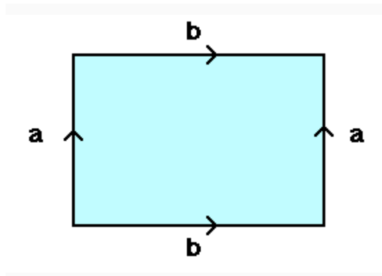
To describe Professor Walker's work, we'll need a little bit of background in algebra. Note that if you add, subtract, and multiply polynomials you get new polynomials. You can't, however, divide them and expect a polynomial in return, as all exponents appearing in polynomials should be positive integers. Such a collection in algebra is called a *ring*, and the set of all polynomials in a given set of variables (say $\{x, y, z\}$) is called a *polynomial ring.* The study of polynomial rings as an object of study goes back at least 130 years to the seminal work of David Hilbert. Two of Hilbert's most famous theorems, the Hilbert Basis Theorem and Hilbert's Syzygy Theorem, both proved in 1890, concern properties of polynomial rings. These results ushered in a new area research in mathematics now known as *commutative algebra* (the word "commutative" here refers to the fact the multiplication is commutative in these rings) and thrives to this day. In fact, over the past two years there have been a remarkable number of breakthroughs on several long-studied problems concerning polynomial rings, with Professor Walker's result being among them.

Before going further, we firm up notation a bit. We'll let $k[x_1, \ldots, x_n]$ denote the polynomial ring in the variables $x_1, \ldots, x_n$ where the coefficients in front of the variables comes from a field k, which for our purposes will be either the real numbers or the complex numbers. A *module* over $k[x_1, \ldots, x_n]$ is the analogue of a vector space, and can be represented as a matrix (not necessarily square) whose entries are elements of $k[x_1, \ldots, x_n]$. One might hope to classify all modules over the ring $k[x_1, \ldots, x_n]$ up to isomorphism, much as the Jordan Canonical Form classifies all matrices (with entries from k) up to similarity. But this is known to be impossible for polynomial rings. Instead, one hopes to understand other invariants of such modules, and first and foremost among these are the *Betti numbers.*

Betti numbers of modules are the algebraic analogues of the invariants of a topological space that go by the same name. In topology, the i-th Betti number $\beta_i(T)$ of a topological space T refers, loosely speaking, to the number of cells of dimension i that are needed to build a space. For example, the representation of the torus



as the result of making edge identifications on a rectangle as in the picture



shows that we need 1 point, 2 edges and 1 two-dimensional piece (the rectangle itself) to build a torus, and the Betti numbers of the torus T are indeed

$$\beta_0(T) = 1, \beta_1(T) = 2, \text{ and } \beta_2(T) = 1.$$

In algebra, the Betti numbers $\beta_0(M), \beta_1(M), \ldots$ of a module M are the ranks of the free modules occurring in its minimal free resolution; that is, they are the smallest natural numbers $\beta_0, \beta_1, \ldots$ such that there exist a exact sequence of modules of the form

$$\cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0,$$

where R^β denotes the direct sum of β copies of $R = k[x_1, \ldots, x_n]$.

As an example, suppose $n = 2$ so that we are talking about modules over the polynomial ring in two variables. Consider the module M over $k[x_1, x_2]$ represented by the matrix $[x_1\ x_2]$. The minimal free resolution of M takes the form

$$0 \to k[x_1, x_2]^1 \to k[x_1, x_2]^2 \to k[x_1, x_2]^1 \to M \to 0$$

so that its Betti numbers are

$$\beta_0(M) = 1, \beta_1(M) = 2, \text{ and } \beta_2(M) = 1.$$

The fact that this list is identical to the list of topological Betti numbers of the torus is no coincidence — there is a very real sense in which the module M we have described here is an algebraic representation of the torus.

The Hilbert Basis Theorem and the Hilbert Syzygy Theorem, mentioned above, imply that

- the free modules occurring in the minimal resolution of M have finite rank, so that the list of Betti numbers of a module really is a list of integers, and
- the list terminates after $n + 1$ steps, where n is the number of variables.

Thus, attached to any module M over the ring $k[x_1, \ldots, x_n]$ is a finite list of positive integers

$$(\beta_0(M), \beta_1(M), \ldots, \beta_n(M)) \in \mathbb{N}^{\times n+1}$$

which describe, roughly, the sizes of the various free modules needed to build M. This stands in analogy (and it is actually more that just an analogy) with the situation in topology, where a compact CW complex of dimension n can be built from finitely many cells of dimensions 0 through n.

There is a special class of modules, called *finite length* modules, which are of particular interest to researchers. In the 1970's David Buchsbaum and David Eisenbud and, independently, Geoffrey Horrocks, formulated a basic question about the smallest possible values of Betti numbers of finite length modules over the ring $k[x_1, \ldots, x_n]$:

BEH Conjecture: *The Betti numbers of a non-zero, finite length module M over the ring $k[x_1, \dots, x_n]$ satisfy $\beta_i(M) \geq \binom{n}{i}$.*

By the Binomial Theorem, $\sum_{i=1}^{n} \binom{n}{i} = 2^n$, and thus the original BEH Conjecture immediately suggests a weak form of it, which is sometimes called the "Total Rank Conjecture", formulated by UNL's own Lucho Avramov in 1985. Professor Avramov promoted the notion that the weak form of the BEH Conjecture is actually the more plausible conjecture, in part because of its relationship with the well-known *Toral Rank Conjecture* in topology.

Total Rank Conjecture: *The sum of the Betti numbers of a non-zero, finite length module M over the ring $k[x_1, \dots, x_n]$ satisfies $\sum_i \beta_i(M) \geq 2^n$.*

There was little progress on either conjecture over the years, and it was thus a great surprise to many when, in December 2016, Professor Walker presented a proof of the Total Rank Conjecture (for almost all fields k, including the reals and complexes) at a conference in Oberwolfach, Germany.

Those attending the joint meeting of the AMS-MAA in January of 2018 can learn more about Walker's result in Craig Huneke's talk "How Complicated are Polynomials in Many Variables", which is part of the Current Events Bulletin Session on Jan. 12.