November 17th, 2011 **Probe II Exam** UNL Math Day

All problems carry equal weight. Marks are awarded for completeness and clarity. A correct answer poorly explained will not earn full marks.

Question 1. Determine all pairs (x, y) of positive integers such that $4x^2 - y^2 = 240$.

Answer: Observe that $4x^2 - y^2 = (2x + y)(2x - y)$ so $2x + y$ and $2x - y$ must be factors of 240. As $2x + y$ and $2x - y$ have the same parity (indeed they each have the same parity as y) they must both be even.

Defining positive integers m and n by $2m = 2x+y$ and $2n = 2x-y$, then $mn = 60$, $m + n = 2x$, and $m - n = y$. Since $m + n$ is even and at least one of m or n is even, both m and n are even.

Defining positive integers m' and n' by $2m' = m$ and $2n' = n$, then $m'n' = 15$ and $m' > n'$. Clearly, the two choices are $m' = 15$, $n' = 1$, which gives $x = m' + n' = 16$ and $y = 2(m' - n') = 28$, and $m' = 5$, $n' = 3$, which gives $x = 8$ and $y = 4$.

Question 2. Divide the positive integers into groups as follows: 1 | 2 3 | 4 5 6 | 7 8 9 10 | 11 . . . What is the sum of the numbers in the k-th group?

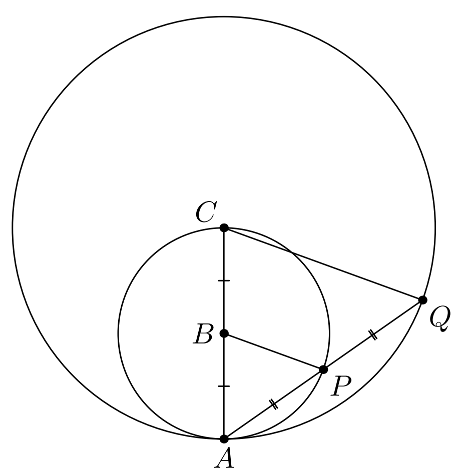
Answer: Recall (or prove) that $1 + 2 + \ldots + (k - 1) = k(k - 1)/2$, so the k^{th} group consists of the numbers

$$\frac{k(k-1)}{2} + 1, \frac{k(k-1)}{2} + 2, \frac{k(k-1)}{2} + 3, \ldots, \frac{k(k-1)}{2} + k.$$

The sum of these numbers (of which there are k) is

$$k\frac{k(k-1)}{2} + \frac{k(k+1)}{2} = \frac{k}{2}(k(k-1) + k + 1) = \frac{k(k^2+1)}{2}.$$

Question 3. Fix a point A on some circle. For each point P on the circle let Q be the point on the ray AP with $AP = PQ$. Determine, with proof, the locus of all these points Q.



Answer: This locus is the circle whose center is the point C diametrically opposite A, with radius equalling the diameter of the original circle. Let B be the center of the original circle. The triangles $\triangle ABP$ and $\triangle ACP$ are similar since they share the angle $\angle PAB$ and $AQ/AP = AC/AB = 2$. We have $BP = BA$ since they are radii of the original circle. By similarity $CQ = CA$, independently of the particular point P chosen.

Question 4. Find the remainder when the polynomial $p(x)$ below is divided by x^2-1, where

$$p(x) = x^{2011} + x^{1869} + x^{1776} + x^{1492} + x^{1216} + x^{1066} + x^{476}.$$

Answer: The quotient $q(x)$ and remainder $r(x)$ when we divide $p(x)$ by x^2-1 satisfy

$$p(x) = (x^2-1)q(x) + r(x), \qquad \deg r(x) < 2.$$

Let's write $r(x) = Ax + B$ for some numbers A, B. Letting $x = \pm 1$,

$$\begin{aligned} p(1) &= A + B \\ p(-1) &= -A + B. \end{aligned}$$

It is easy to evaluate $p(1)$; each term contributes 1. In $p(-1)$ each even power contributes 1 and each odd power contributes -1. Thus $A + B = 7$, $-A + B = 3$, with solution $A = 2, B = 5$. Hence $r(x) = 2x + 5$.

Question 5. In a triangle $\triangle ABC$ the sides a, b, c (here a is the length of the side opposite vertex A, etc.) satisfy $(a+b+c)(a+b-c) = 3ab$. What is the angle at C?

Answer: Expanding the difference of squares, $(a+b)^2 - c^2 = 3ab$ and so $c^2 = a^2+b^2-ab$. If θ is the angle at C, then the Cosine Law gives $c^2 = a^2+b^2-2ab\cos\theta$ and so $\cos\theta = 1/2$. As θ is between 0 and π, $\theta = \pi/3$.

Question 6. A *variegated word* is a string of letters that contains no pair of repeated letters adjacent to one another, nor any pattern like $\ldots L \ldots M \ldots L \ldots M \ldots$ where L and M are any two (different) letters. For instance SCENES is variegated, but SENSIBLE is not, since it contains ...S ...E ...S ...E .... Prove that every variegated word has some letter that appears exactly once.

Answer: Suppose that w is a variegated word in which each letter that appears in w does so at least twice. Consider a pair of repeated letters $\ldots L \ldots L \ldots$ that appear at minimum distance from one another in w. Since they are not adjacent (w is variegated) some other letter, say M, appears between them: $\ldots L \ldots M \ldots L \ldots$ There is at least one other occurrence of M in w by hypothesis, and since w is variegated it can't appear to the left of the first L nor to the right of the second. Thus there must be another M between the L's: $\ldots L \ldots M \ldots M \ldots L \ldots$ This contradicts the choice of L, and so no such word w exists.