Question 1. What is the value of

$$\left(1-\frac{1}{2}\right) \times \left(1-\frac{1}{3}\right) \times \left(1-\frac{1}{4}\right) \times \cdots \times \left(1-\frac{1}{2012}\right) ?$$

(A) $\dfrac{2012}{2013}$ (B) $\dfrac{1}{2012!}$ (C) $\dfrac{1}{2013!}$ (D) $1-\dfrac{1}{2012!}$ $\boxed{\text{E}}$ $\dfrac{1}{2012}$

Answer. Writing each factor as a fraction we get

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2012}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{2011}{2012} = \frac{1}{2012}.$$

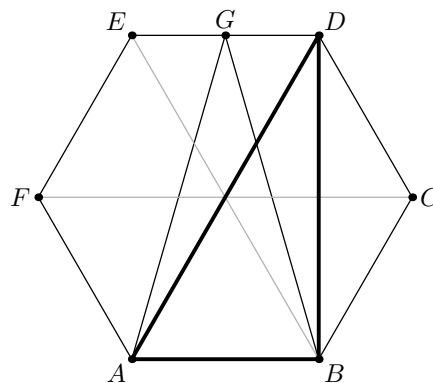**Question 2.** If $xy = 2$, $yz = 3$ and $xz = 5$, what is the value of $x^2 + y^2 + z^2$?

(A) 20   (B) $\dfrac{6}{5}$ (C) $\dfrac{65}{6}$ $\boxed{\text{D}}$ $\dfrac{361}{30}$ (E) 32

Answer. We have $x^2 = (xy)(xz)/(yz) = 2 \times 5/3$. Similarly $y^2 = 2 \times 3/5$, $z^2 = 3 \times 5/2$ and

$$x^2 + y^2 + z^2 = \frac{10}{3} + \frac{6}{5} + \frac{15}{2} = \frac{100 + 36 + 225}{30} = \frac{361}{30}$$

Question 3. In the diagram below point G is the midpoint of the segment DE. What fraction of the area of the regular hexagon $ABCDEF$ is the area of $\triangle AGB$?

(A) $\dfrac{1}{4}$ $\boxed{\text{B}}$ $\dfrac{1}{3}$ (C) $\dfrac{3}{8}$ (D) $\dfrac{5}{12}$ (E) $\dfrac{1}{2}$

Answer. Divide the hexagon into six triangles from the center as in the diagram. The area of $\triangle AGB$ is the same as the area of $\triangle ADB$ since they have the same base and height. The area of $\triangle ADB$ is clearly the same as two of the small triangles, whereas the hexagon contains all six. Thus the ratio is $1/3$.

Question 4. A cube of side 4cm is made up of individual 1 cm cubes. How many of these 1 cm cubes are face to face with exactly four other 1 cm cubes?

(A) 0   (B) 8   (C) 16 $\boxed{\text{D}}$ 24   (E) 28

Answer. The only small cubes that have four face-to-face neighbors are those along the edges of the larger cube. There are two such cubes per edge of the large cube, so there are 24 in all.

Question 5. A circle with center B and radius 5 is tangent to the sides of an angle of $60°$. A larger circle with center C is tangent to the sides of the angle and to the first circle. What is the radius of the larger circle?

(A) $5\sqrt{3}$ (B) $10 + 5\sqrt{3}$ (C) 10 $\boxed{\text{D}}$ 15   (E) $20 - 5\sqrt{3}$
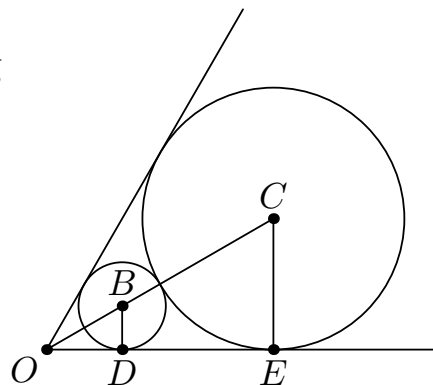
Answer. We have $DB = 5$ and, since $\angle DOB = 30°$, $OB = 10$. If x is the radius of the larger circle then $1/2 = \sin(30°) = x/(15 + x)$, hence $x = 15$.

Question 6. Steve buys p oranges. He squeezes $q\%$ of the oranges to make fresh orange juice. How many oranges are left?

(A) $\dfrac{pq}{100}$
$\boxed{\text{B}}$ $\dfrac{100p - pq}{100}$
(C) $\dfrac{100pq}{100}$
(D) $\dfrac{p - q}{100}$
(E) $\dfrac{100q - pq}{100}$

Answer. $(100 - q)\%$ of the oranges remain unsqueezed. This is

$$\frac{100 - q}{100} p = \frac{100p - qp}{100} \text{ oranges.}$$

Question 7. The sum of the fourth and fifth terms in an increasing sequence of consecutive integers is 69. What is the sum of the first five terms of the sequence?

(A) 95 (B) 96 $\boxed{\text{C}}$ 165 (D) 255 (E) Not enough information

Answer. Let a be the first term, so that the sequence starts $a, a + 1, a + 2, a + 3, a + 4, \ldots$. Since $2a + 7 = (a + 3) + (a + 4) = 69$ we know $a = 31$, so the sum of the first five terms is $5a + 10 = 165$.

Question 8. One coin is picked randomly from two coins, and that coin is then tossed three times. It landed heads up each time. If one of the two original coins was two-headed and the other a fair coin, what is the probability that the chosen coin was the two-headed coin?

$\boxed{\text{A}}$ $\dfrac{8}{9}$
(B) $\dfrac{7}{8}$
(C) $\dfrac{2}{3}$
(D) $\dfrac{1}{2}$
(E) $\dfrac{1}{3}$

Answer. There are 16 equally likely outcomes for the choice of coins and subsequent coin flips. Of these, 9 result in heads each time: all outcomes where we picked the two-headed coin, and one where we picked the fair coin. Of these 9 outcomes 8 of them involved the two-headed coin, so the probability is $8/9$.

Question 9. In the figure, the circle with center O passes through A and B and BD is tangent to the circle. If $\angle AOB = 70°$ then what is $\angle CBD$?

(A) 30° $\boxed{\text{B}}$ 35° (C) 40° (D) 45° (E) 50°
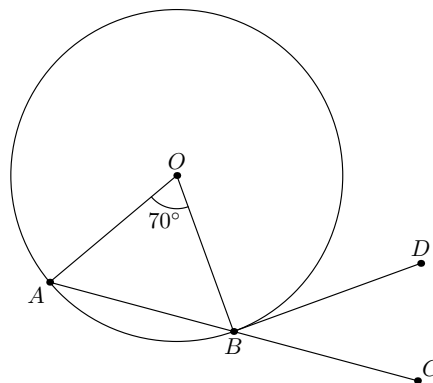
Answer. Since $\triangle AOB$ is isosceles we know $\angle OAB = \angle OBA = (180 - 70)/2 = 55°$. Now, together, angles $\angle OBA$, $\angle DBO$, and $\angle CBD$ make up 180°. Since OB is a radius of the circle and DB is a tangent we know that $\angle DBO = 90°$. Thus $\angle CBD = 180 - 90 - 55 = 35°$.

Question 10. Suppose that x and y are two positive real numbers such that $\log_x y + \log_y x = 2$. What does $\left(\log_x y\right)^3 + \left(\log_y x\right)^3$ equal?

$\boxed{\text{A}}$ 2 (B) 6 (C) 8 (D) 9 (E) Not enough information

Answer. First note that $\log_y x = 1/\log_x y$. Therefore if we write $a = \log_x y$ then $a + \frac{1}{a} = 2$. Thus

$$8 = \left(a + \frac{1}{a}\right)^3 = a^3 + 3a + 3\frac{1}{a} + \frac{1}{a^3}$$

$$= a^3 + \frac{1}{a^3} + 6.$$

Hence $a^3 + 1/a^3 = 2$.

2

Question 11. A cube has edges of length 6. One of its square faces is $ABCD$ and E is the center of the cube. What is the volume of the pyramid $ABCDE$?

(A) 54 $\boxed{\text{B}}$ 36   (C) 108   (D) 72   (E) 216

Answer. The volume of a pyramid is $\frac{1}{3}Ah$ where A is the area of its base and h is its height. The area of the bases is 36 and the height is 3, so the volume is $\frac{1}{3}36 \times 3 = 36$.

Question 12. Which of the following numbers is the largest?

(A) 6^{100} (B) 5^{200} (C) 4^{300} $\boxed{\text{D}}$ 3^{400} (E) 2^{500}

Answer. The relative size of these numbers are the same as the relative sizes of their 100th roots, which are respectively $6^1 = 6$, $5^2 = 25$, $4^3 = 64$, $3^4 = 81$, and $2^5 = 32$.

Question 13. A car travels up a hill at an average speed of 50 miles per hour. At what average speed does the car need to travel the same distance back down the hill to average 60 miles per hour for the entire trip?

(A) 65   (B) 68   (C) 70 $\boxed{\text{D}}$ 75   (E) Not enough information provided

Answer. Suppose that the distance up the hill is d and the desired average speed is s. Going up the hill took time $d/50$, coming down takes time d/s, and the total of these should be $2d/60$. Thus $d/50 + d/s = 2d/60$. Dividing through by d and multiplying through by $50s$, we get

$$s + 50 = \frac{50}{30}s, \quad \text{and therefore} \quad \frac{2}{3}s = 50, \quad \text{i.e.,} \quad s = 75.$$

Question 14. If $\dfrac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}} = 2012$, what is the value of $\dfrac{x - y}{x + y}$?

$\boxed{\text{A}}$ $-\dfrac{1}{2012}$ (B) $\dfrac{1}{2012}$ (C) -2012 (D) 2012   (E) Not enough information
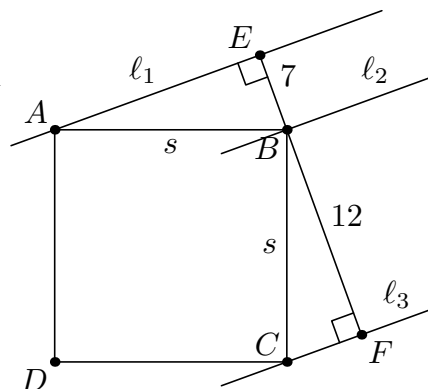
Answer. Multiplying top and bottom by xy we have

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}} = \frac{y + x}{y - x} = 2012.$$

Thus $\dfrac{x - y}{x + y} = -\dfrac{1}{2012}$.

Question 15. Let $ABCD$ be a square. Three parallel lines ℓ_1, ℓ_2 and ℓ_3 are drawn through the vertices A, B and C respectively. The distance between ℓ_1 and ℓ_2 is 7, and the distance between ℓ_1 and ℓ_3 is 19. What is the area of the square $ABCD$?

(A) 49   (B) 144 $\boxed{\text{C}}$ 193   (D) 227   (E) Not enough information

Answer. In the diagram we have that angles $\angle EBA$ and $\angle FBC$ are complementary, as are $\angle EBA$ and $\angle EAB$. Thus $\angle EAB = \angle FBC$. We see that triangles $\triangle EAB$ and $\triangle FBC$ are congruent by AAAS. Thus,

$$\text{area} = s^2 = 7^2 + 12^2 = 193.$$

3

Question 16. If $\sin^2 x - \tan^2 x = -\cos^2 x$, then what is the value of $\sec x$?

(A) 0   (B) ± 1 $\boxed{\text{C}}$ $\pm\sqrt{2}$ (D) $\pm\sqrt{3}$ (E) Not enough information

Answer. Since $\sin^2 x - \tan^2 x = -\cos^2 x$ we have

$$\sec^2 x = 1 + \tan^2 x = 1 + \sin^2 x + \cos^2 x = 2, \quad \text{so} \quad \sec x = \pm\sqrt{2}.$$

Question 17. What is the sum of the digits in the number N, where

$$N = 11 \times 101 \times 10,001 \times 100,000,001 \times 10,000,000,000,000,001 ?$$

(A) 1   (B) 15   (C) 16   (D) 31 $\boxed{\text{E}}$ 32

Answer. First note that if we multiply a string of ones $1111\cdots1$ of length n by a number of the form $100\ldots001$ with $n-1$ zeroes, then the result is a string of ones of length $2n$. This process is occurring as we compute this product from left to right. 11 becomes 1111 becomes 11111111 becomes.... The final result is a string of 32 ones. Thus the sum of digits of N is 32.

Question 18. The smallest positive integer n for which the decimal expansion of $n!$ ends in 3 zeroes is:

(A) 10   (B) 12   (C) 14 $\boxed{\text{D}}$ 15   (E) 16

Answer. For $n!$ to end in three zeroes it must be divisible by $1000 = 5^3 \times 2^3$. For there to be three factors of 5 we must have $n \geq 15$, and at this point there are plenty of factors of 2. Thus the smallest n is $n = 15$.

Question 19. The solution sets of the polynomial equations $f(x) = 0$ and $g(x) = 0$ are, respectively, $\{0,1,2,3\}$ and $\{2,3,4\}$. How many different numbers are in the solution set of the polynomial equation $f(x)g(x) = 0$?

(A) 4 $\boxed{\text{B}}$ 5   (C) 6   (D) 7   (E) 12

Answer. We have $f(x)g(x) = 0$ if either $f(x) = 0$ or $g(x) = 0$, i.e., if x is in the union of the sets $\{0,1,2,3\}$ and $\{2,3,4\}$. Thus the solution set of $f(x)g(x) = 0$ is $\{0,1,2,3,4\}$, with 5 elements.

Question 20. The sum of all real numbers x that satisfy the equation $\left(x^2 - 5x + 5\right)^{x^2+4x-60} = 1$ is:

(A) -1 $\boxed{\text{B}}$ 1   (C) 3   (D) 5   (E) 7

Answer. We know that $a^b = 1$ exactly if $a = 1$ or $b = 0$. If $x^2 - 5x + 5 = 1$ then $x^2 - 5x + 4 = (x - 4)(x - 1) = 0$ so $x = 1, 4$. If $x^2 + 4x - 60 = (x - 6)(x + 10) = 0$ then $x = 6, -10$. The sum of all solutions is $1 + 4 + 6 - 10 = 1$.

Question 21. A line with slope 2 intersects a line with slope 6 at the point $(20, 12)$. What is the distance between the x-intercepts of these lines?

(A) 1   (B) 2   (C) 3 $\boxed{\text{D}}$ 4   (E) 5

Answer. If $(x, 0)$ is on the line with slope 2 through $(20, 12)$ then $(12 - 0)/(20 - x) = 2$, so $x = 14$. Similarly the x-intercept for the line of slope 6 satisfies $(12 - 0)/(20 - x) = 6$, i.e., $x = 18$. The points $(14, 0)$ and $(18, 0)$ are distance 4 from one another.

Question 22. How many different pairs or real numbers (x, y) satisfy the following system of equations:

$$\begin{cases} |x - y| = 1 \\ \frac{x}{y} = xy. \end{cases}$$

(A) 0   (B) 1   (C) 2   (D) 3 $\boxed{\text{E}}$ 4

Answer. If $x = 0$ then the second equation is satisfied automatically and from the first equation $y = \pm 1$. This gives us pairs $(0, 1)$ and $(0, -1)$. If $x \neq 0$ then we can cancel the x's in the second equation and get $1/y = y$, i.e. $y = \pm 1$. Now from the first equation we get $x = y \pm 1$, so we get additional pairs $(2, 1)$, and $(-2, -1)$. Thus there are a total of 4 pairs satisfying both equations.

Question 23. Two circles with centers A and B have radii of 9 cm. and 12 cm., respectively. The centers are 28 cm. apart. Point C is the intersection of AB with a common internal tangent to the circles. How far is A from C?

(A) 9 cm $\boxed{\text{B}}$ 12 cm   (C) 13 cm   (D) 15 cm   (E) 20 cm
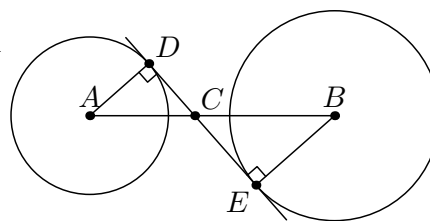
Answer. Let $x = AC$ and $y = BC$ be the distances from the intersection point to the centers of the circles. We know $x + y = 28$ and, since triangles $\triangle ACD$ and $\triangle BCE$ are similar,

$$\frac{x}{9} = \frac{y}{12}, \quad \text{hence} \quad 4x = 3y.$$

The unique solution to this pair of equations is $x = 12$, $y = 16$, so $AC = 12$.

Question 24. Bob had an average score of 85 on his first 8 quizzes. His average on his first 9 quizzes was 86. What was his score on the ninth quiz?

(A) 86   (B) 88   (C) 90   (D) 92 $\boxed{\text{E}}$ 94

Answer. The total score on his first 8 quizzes is $85 \times 8 = 680$. The total score on his first 9 quizzes is $86 \times 9 = 774$. Thus the score on his 9th quiz is $774 - 680 = 94$.

Question 25. For a tetrahedron $ABCD$, a plane P is called a *middle plane* if all four distances from the vertices A, B, C and D to the plane P are equal. How many middle planes does a given tetrahedron have?

(A) 10 $\boxed{\text{B}}$ 7   (C) 6   (D) 5   (E) 4

Answer. For any given vertex there is a plane parallel to the opposite face that is the same distance from all the vertices and separates the given vertex from the opposite face. This gives 4 middle planes. In addition, for any pair of opposite edges there is a plane parallel to both edges that is the same distance from all the vertices and seperates the edges from one another. Since there are 3 pairs of opposite edges this gives another 3 middle planes, for a total of 7.