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Oscillation of an Euler–Cauchy Dynamic Equation

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Abstract. The Euler-Cauchy differential equation and difference equation are well known. Here we study a more general Euler-Cauchy dynamic equation. For this more general equation when we have complex roots of the corresponding characteristic equation we for the first time write solutions of this dynamic equation in terms of a generalized exponential function and generalized sine and cosine functions. This result is even new in the difference equation case. We then spend most of our time studying the oscillation properties of the Euler-Cauchy dynamic equation. Several oscillation results are given and an open problem is posed.

1. **Introduction.** In this paper we will assume that the reader is familiar with the elementary concepts and notation used in the calculus on time scales (see, for example, Bohner and Peterson [2]). We are concerned with the so-called Euler–Cauchy dynamic equation

$$\sigma(t)tx^{\Delta\Delta} + atx^{\Delta} + bx = 0, \tag{1}$$

on a time scale $\mathbb{T}$ (closed subset of the reals $\mathbb{R}$), where we assume $t_0 = \inf \mathbb{T} > 0$. We will assume throughout the regressivity condition

$$\sigma(t)t - at\mu(t) + b\mu^2(t) \neq 0 \tag{2}$$

for $t \in \mathbb{T}^\kappa$. The equation

$$\lambda^2 + a\lambda + b = 0 \tag{3}$$

is called the characteristic equation of the Euler–Cauchy dynamic equation (1) and the roots of (3) are called the characteristic roots of (1). We now give an alternate shorter proof of Theorems 3.63 and 3.66 in [2]. (Our proof combines the proofs of these two theorems in a novel way.)

Theorem 1. *Assume λ_1, λ_2 are solutions of the chacteristic equation (3). If $\lambda_1 \neq \lambda_2$, then*

$$x(t) = c_1 e_{\frac{\lambda_1}{t}}(t, t_0) + c_2 e_{\frac{\lambda_2}{t}}(t, t_0)$$

is a general solution of (1). If $\lambda_1 = \lambda_2$, then

$$x(t) = c_1 e_{\frac{\lambda_1}{t}}(t, t_0) + c_2 e_{\frac{\lambda_1}{t}}(t, t_0) \int_{t_0}^{t} \frac{1}{s + \lambda_1\mu(s)} \Delta s$$

is a general solution of (1).

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Proof. Note if we let D denote the Delta–derivative operator with respect to t, then the Euler–Cauchy dynamic equation (1) can be written in the factored form

$$(tD-\lambda_2)(tD-\lambda_1)x=0. \tag{4}$$

This factorization was noted by Akin–Bohner and Bohner ([1] and [3], Chapter 2) when they discovered how to define the n-th order Euler–Cauchy dynamic equation. Assume that x is a solution of the dynamic equation (4) and let

$$y=(tD-\lambda_1)x, \tag{5}$$

then by (4), y is a solution of

$$(tD-\lambda_2)y=0,$$

which is equivalent to the dynamic equation

$$y^\Delta=\frac{\lambda_2}{t}y. \tag{6}$$

The regressivity condition (2) ensures that $\frac{\lambda_1}{t},\frac{\lambda_2}{t}\in\mathcal{R}$, where $\mathcal{R}$ is the regressive group [2], page 58. Hence the dynamic equation (6) is regressive and so we get

$$y(t)=c_2e_{\frac{\lambda_2}{t}}(t,t_0).$$

It follows from (5) that x satisfies the dynamic equation

$$(tD-\lambda_1)x=c_2e_{\frac{\lambda_2}{t}}(t,t_0),$$

or equivalently

$$(D-\frac{\lambda_1}{t})x=c_2\frac{1}{t}e_{\frac{\lambda_2}{t}}(t,t_0). \tag{7}$$

Using the variation of constants formula [2], page 77, we get that

$$\begin{aligned}
x(t) &= c_1e_{\frac{\lambda_1}{t}}(t,t_0)+c_2\int_{t_0}^{t}e_{\frac{\lambda_1}{t}}(t,\sigma(s))\left(\frac{1}{s}e_{\frac{\lambda_2}{t}}(s,t_0)\right)\Delta s\\
&= c_1e_{\frac{\lambda_1}{t}}(t,t_0)+c_2e_{\frac{\lambda_1}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{s}e_{\frac{\lambda_1}{t}}(t_0,\sigma(s))e_{\frac{\lambda_2}{t}}(s,t_0)\Delta s\\
&= c_1e_{\frac{\lambda_1}{t}}(t,t_0)+c_2e_{\frac{\lambda_1}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{s}e_{\ominus\frac{\lambda_1}{t}}(\sigma(s),t_0)e_{\frac{\lambda_2}{t}}(s,t_0)\Delta s\\
&= c_1e_{\frac{\lambda_1}{t}}(t,t_0)+c_2e_{\frac{\lambda_1}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{s+\lambda_1\mu(s)}e_{\ominus\frac{\lambda_1}{t}}(s,t_0)e_{\frac{\lambda_2}{t}}(s,t_0)\Delta s\\
&= c_1e_{\frac{\lambda_1}{t}}(t,t_0)+c_2e_{\frac{\lambda_1}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{s+\lambda_1\mu(s)}e_{\frac{\lambda_2}{t}\ominus\frac{\lambda_1}{t}}(s,t_0)\Delta s.
\end{aligned}$$

First note that if $\lambda_1=\lambda_2$, then we get the desired result that

$$x(t)=c_1e_{\frac{\lambda_1}{t}}(t,t_0)+c_2e_{\frac{\lambda_1}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{s+\lambda_1\mu(s)}\Delta s.$$

Next assume that $\lambda_1\neq\lambda_2$, then using the formula

$$\int_{t_0}^{t}\frac{1}{s+\lambda_1\mu(s)}e_{\frac{\lambda_2}{t}\ominus\frac{\lambda_1}{t}}(s,t_0)\Delta s=\frac{1}{\lambda_2-\lambda_1}\left[e_{\frac{\lambda_2}{t}\ominus\frac{\lambda_1}{t}}(t,t_0)-1\right]$$

we are led to the final result. □

Next we would like to show that if our characteristic roots are complex, then there is a nice form for all real-valued solutions of the Euler–Cauchy dynamic equation in terms of the generalized exponential and trigonometric functions. Even in the difference equations case the complex roots are not considered (see Kelley and Peterson [7]).

Theorem 2. *Assume that the characteristic roots of* (1) *are complex, that is* $\lambda_{1,2} = \alpha \pm i\beta$*, where* $\beta > 0$*, and* $\frac{\alpha}{t}, \frac{\beta}{t+\alpha\mu(t)} \in \mathcal{R}$*. Then*

$$x(t) = c_1 e_{\frac{\alpha}{t}}(t,t_0)\cos_{\frac{\beta}{t+\alpha\mu(t)}}(t,t_0) + c_2 e_{\frac{\alpha}{t}}(t,t_0)\sin_{\frac{\beta}{t+\alpha\mu(t)}}(t,t_0)$$

is a general solution of the Euler–Cauchy dynamic equation (1).

Proof. Assume $\lambda_{1,2} = \alpha \pm i\beta$, where $\beta > 0$, are the characteristic roots. Then by Theorem 1,

$$e_{\frac{\alpha+i\beta}{t}}(t,t_0), \quad e_{\frac{\alpha-i\beta}{t}}(t,t_0)$$

are solutions of (1). We want to find $\tilde{\beta}$ so that

$$\frac{\alpha}{t} + i\frac{\beta}{t} = \frac{\alpha}{t} \oplus i\frac{\tilde{\beta}}{t}. \tag{8}$$

Solving this equation we get

$$\frac{\tilde{\beta}}{t} = \frac{\beta}{t+\alpha\mu(t)}. \tag{9}$$

Hence if $\tilde{\beta}$ is defined by (9), then (8) holds. Similarly

$$\frac{\alpha}{t} - i\frac{\beta}{t} = \frac{\alpha}{t} \oplus (-i)\frac{\tilde{\beta}}{t}.$$

It follows that

$$\begin{aligned} x_1(t) &= \frac{1}{2}e_{\frac{\alpha+i\beta}{t}}(t,t_0) + \frac{1}{2}e_{\frac{\alpha-i\beta}{t}}(t,t_0) \\ &= \frac{1}{2}e_{\frac{\alpha}{t}\oplus i\frac{\tilde{\beta}}{t}}(t,t_0) + \frac{1}{2}e_{\frac{\alpha}{t}\oplus(-\frac{i\tilde{\beta}}{t})}(t,t_0) \\ &= e_{\frac{\alpha}{t}}(t,t_0)\left(\frac{e_{i\frac{\tilde{\beta}}{t}}(t,t_0) + e_{-i\frac{\tilde{\beta}}{t}}(t,t_0)}{2}\right) \\ &= e_{\frac{\alpha}{t}}(t,t_0)\cos_{\frac{\tilde{\beta}}{t}}(t,t_0) \\ &= e_{\frac{\alpha}{t}}(t,t_0)\cos_{\frac{\beta}{t+\alpha\mu(t)}}(t,t_0) \end{aligned}$$

is a solution. Similarly

$$x_2(t) = e_{\frac{\alpha}{t}}(t,t_0)\sin_{\frac{\beta}{t+\alpha\mu(t)}}(t,t_0)$$

is a solution. Since x_1, x_2 are linearly independent solutions on $\mathbb{T}$ we get the desired result. □

2. **Oscillation Results.** In this section we will be concerned with the oscillation of the Euler–Cauchy dynamic equation (1). We assume throughout this section that $\mathbb{T}$ is now unbounded above. We now show if the characteristic roots of (1) are complex how a general solution can be written in terms of the classical exponential function and classical trigonometric functions.

Lemma 1. *If the characteristic roots are complex, that is $\lambda_{1,2} = \alpha \pm i\beta$, where $\beta > 0$, then*

$$x(t) = A(t)\left(c_1 \cos B(t) + c_2 \sin B(t)\right),$$

where

$$A(t) = e^{\int_{t_0}^{t} \Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau}, \quad B(t) = \int_{t_0}^{t} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau \tag{10}$$

is a general solution of the Euler–Cauchy dynamic equation (1). *If, in addition, every point in $\mathbb{T}$ is isolated, then for $t \in \mathbb{T}$,*

$$A(t) = \prod_{\tau=t_0}^{\rho(t)} \frac{1}{\tau}\sqrt{((\tau + \mu(\tau)\alpha)^2 + \beta^2\mu^2(t))}, \quad B(t) = \sum_{\tau=t_0}^{\rho(t)} Arctan\left(\frac{\beta\mu(\tau)}{\tau + \alpha\mu(\tau)}\right).$$

Proof. Note that (see page 59, Bohner and Peterson [2]) the generalized exponential

$$\begin{aligned} e_{\frac{\alpha+i\beta}{t}}(t, t_0) &= e^{\int_{t_0}^{t} \xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\Delta\tau} \\ &= e^{\int_{t_0}^{t} \Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right) + i\Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau} \\ &= A(t)e^{iB(t)} \\ &= A(t)\left(\cos B(t) + i \sin B(t)\right). \end{aligned}$$

It follows that the imaginary part and real part

$$x_1(t) := A(t)\cos B(t), \quad x_2(t) := A(t)\sin B(t)$$

are solutions of (1). Since they can be shown to be linearly independent on $\mathbb{T}$ the result follows.
Now assume that every point in $\mathbb{T}$ is isolated, then

$$\begin{aligned} &\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right) \\ =\ & \frac{1}{\mu(\tau)} \operatorname{Log}\left(1 + \mu(\tau)\frac{\alpha+i\beta}{\tau}\right) \\ =\ & \frac{1}{\mu(\tau)} \log\left|\frac{\tau + \alpha\mu(\tau)}{\tau} + i\frac{\beta\mu(\tau)}{\tau}\right| + \frac{i}{\mu(\tau)} Arg\left(\frac{\tau + \alpha\mu(\tau)}{\tau} + i\frac{\beta\mu(\tau)}{\tau}\right) \\ =\ & \frac{1}{\mu(\tau)} \log\left(\frac{1}{\tau}\sqrt{(\tau + \alpha\mu(\tau))^2 + \beta^2\mu^2(\tau)}\right) + \frac{i}{\mu(\tau)} Arctan\left(\frac{\beta\mu(\tau)}{\tau + \alpha\mu(\tau)}\right). \end{aligned}$$

Hence

$$\Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right) = \frac{1}{\mu(\tau)} \log\left(\frac{1}{\tau}\sqrt{(\tau + \alpha\mu(\tau))^2 + \beta^2\mu^2(\tau)}\right) \tag{11}$$

and

$$\Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right) = \frac{1}{\mu(\tau)} Arctan\left(\frac{\beta\mu(\tau)}{\tau + \alpha\mu(\tau)}\right). \tag{12}$$

It follows from (10) and (11) that

$$\begin{aligned} A(t) &= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} \log\left(\frac{1}{\tau}\sqrt{(\tau+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right)\Delta\tau} \\ &= e^{\sum_{\tau=t_0}^{\rho(t)} \log\left(\frac{1}{\tau}\sqrt{(\tau+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right)} \\ &= \prod_{\tau=t_0}^{\rho(t)} \left(\frac{1}{\tau}\sqrt{(\tau+\alpha\mu(\tau))^2+\beta^2\mu^2(\tau)}\right). \end{aligned}$$

It follows from (10) and (12) that

$$\begin{aligned} B(t) &= \int_{t_0}^{t} \frac{1}{\mu(\tau)} Arctan\left(\frac{\beta\mu(\tau)}{\tau+\alpha\mu(\tau)}\right)\Delta\tau \\ &= \sum_{\tau=t_0}^{\rho(t)} Arctan\left(\frac{\beta\mu(\tau)}{\tau+\alpha\mu(\tau)}\right), \end{aligned}$$

which is the desired result. □

Definition 1. If the characteristic roots of (1) are complex, then we say the Euler–Cauchy dynamic equation (1) is oscillatory iff $B(t)$ is unbounded.

As a well-known example note that if $\mathbb{T}$ is the real interval $[1,\infty)$ and the Euler–Cauchy equation has complex roots, then the Euler–Cauchy equation is oscillatory. This follows from what we said here because in this case by (10)

$$B(t) = \beta \int_1^t \frac{1}{\tau} d\tau = \beta \log t$$

which is unbounded. If $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$, then by Lemma 1

$$\begin{aligned} B(t) &= \sum_{\tau=1}^{\rho(t)} Arctan\left(\frac{\beta\mu(t)}{\tau+\alpha\mu(\tau)}\right) \\ &= \sum_{\tau=1}^{\rho(t)} Arctan\left(\frac{\beta(q-1)}{1+\alpha(q-1)}\right) \\ &= \left(\frac{t-1}{q-1}\right) Arctan\left(\frac{\beta(q-1)}{1+\alpha(q-1)}\right), \end{aligned}$$

which is unbounded and hence the Euler–Cauchy dynamic equation on $\mathbb{T} = q^{\mathbb{N}_0}$ is oscillatory when the characteristuc roots are complex. If $\mathbb{T} = \mathbb{N}$, then

$$B(t) = \sum_{k=1}^{t-1} Arctan\left(\frac{\beta}{k+\alpha}\right),$$

which can be shown to be unbounded and hence the Euler–Cauchy dynamic equation on $\mathbb{T} = \mathbb{N}$ is oscillatory when the characteristuc roots are complex. These last two examples where shown in Bohner and Saker [4], Erbe, Peterson, and Saker [6], and Erbe and Peterson [5], but here we established these results directly.

Theorem 3 (Comparison Theorem)**.** *Let $\mathbb{T}_1 := \{t_0, t_1, \cdots\}$ and $\mathbb{T}_2 := \{s_0, s_1, \cdots\}$, where $\{t_n\}$ and $\{s_n\}$ are strictly increasing sequences of positive numbers with limit ∞. If the Euler–Cauchy equation (1) on $\mathbb{T}_1$ is oscillatory and $-\alpha < \frac{s_n}{\mu(s_n)} \leq \frac{t_n}{\mu(t_n)}$, for $n \geq 0$, then the Euler–Cauchy equation (1) on $\mathbb{T}_2$ is oscillatory.*

Proof. Since $\frac{s_n}{\mu(s_n)} \leq \frac{t_n}{\mu(t_n)}$, for $n \geq 0$, we have that

$$\frac{s_n}{\mu(s_n)} + \alpha \leq \frac{t_n}{\mu(t_n)} + \alpha$$

for $n \geq 0$, and therefore (using $-\alpha < \frac{s_n}{\mu(s_n)}$)

$$0 < \frac{s_n + \mu(s_n)\alpha}{\mu(s_n)} \leq \frac{t_n + \mu(t_n)\alpha}{\mu(t_n)}$$

for $n \geq 0$. Taking reciprocals and multiplying by β we obtain

$$\frac{\beta\mu(s_n)}{s_n + \mu(s_n)\alpha} \geq \frac{\beta\mu(t_n)}{t_n + \mu(t_n)\alpha}.$$

This implies that

$$Arctan\left(\frac{\beta\mu(s_n)}{s_n + \alpha\mu(s_n)}\right) \geq Arctan\left(\frac{\beta\mu(t_n)}{t_n + \alpha\mu(t_n)}\right)$$

for $n \geq 0$. This implies that

$$\begin{aligned} B_2(s) = B_2(s_n) &:= \sum_{k=0}^{n-1} Arctan\left(\frac{\beta\mu(s_k)}{s_k + \alpha\mu(s_k)}\right) \\ \geq B_1(t) = B_1(t_n) &:= \sum_{k=0}^{n-1} Arctan\left(\frac{\beta\mu(t_k)}{t_k + \alpha\mu(t_k)}\right). \end{aligned}$$

Since we are assuming that the Euler–Cauchy equation (1) is oscillatory on $\mathbb{T}_1$ we get that $\lim_{n\to\infty} B_1(s_n) = \infty$ and therefore from the above inequality $\lim_{n\to\infty} B_2(t_n) = \infty$, which implies that the Euler–Cauchy equation (1) is oscillatory on $\mathbb{T}_2$. □

Theorem 4. *Assume every point in the time scale $\mathbb{T}$ is isolated and $\lim_{t\to\infty} \frac{t}{\mu(t)}$ exists as a finite number, then the Euler–Cauchy equation in the complex characteristic roots case is oscillatory on $\mathbb{T}$.*

Proof. In this case

$$\begin{aligned} \lim_{t\to\infty} B(t) &= \sum_{\tau=t_0}^{\infty} Arctan\left(\frac{\beta\mu(\tau)}{\tau + \mu(\tau)\alpha}\right) \\ &= \sum_{\tau=t_0}^{\infty} Arctan\left(\frac{\beta}{\frac{\tau}{\mu(\tau)} + \alpha}\right). \end{aligned}$$

It follows that $B(t)$ is unbounded and hence the Euler–Cauchy equation in the complex characteristic roots case is oscillatory on $\mathbb{T}$ □

Theorem 4 does not cover the case when $\mathbb{T}$ is a time scale where $\lim_{t\to\infty} \frac{t}{\mu(t)} = \infty$. The next theorem considers a time scale where $\lim_{t\to\infty} \frac{t}{\mu(t)} = \infty$.

Theorem 5. *Let $p \geq 0$ and let $\mathbb{T}_p := \{t_n : t_0 = 1, t_{n+1} = t_n + \frac{1}{t_n^p}, n \in \mathbb{N}_0\}$. In the complex characteristic roots case, the Euler–Cauchy dynamic equation (1) is oscillatory on $\mathbb{T}_p$.*

Proof. If $p = 0$, then $\mathbb{T} = \mathbb{N}$ and the result was proved earlier in this paper. Assume $p > 0$. Since

$$t_{n+1} = t_n + \frac{1}{t_n^p}, \tag{13}$$

for $n \in \mathbb{N}_0$, the sequence $\{t_n\}$ is strictly increasing. Assume $\lim_{n\to\infty} t_n = L$, where L is a positive constant. Then from (13) we get

$$L = L + \frac{1}{L^p}$$

and this implies that $0 = \frac{1}{L^P}$, which is a contradiction, which proves that the sequence $\{t_n\}$ is unbounded above. Now for $k \in \mathbb{N}_0$ pick N_k so that t_{N_k} is the smallest element in $\mathbb{T}$ that is in the real interval $[k, k+1)$. Then for $N_k \leq j \leq N_{k+1} - 1$,

$$k \leq y_j < k+1.$$

Note that $N_{k+1} - N_k$ is the number of elements of $\mathbb{T}$ in the real interval $[k, k+1)$. Since

$$\frac{1}{k+1} < \frac{1}{y_j} \leq \frac{1}{k}$$

and since $y_{j+1} = y_j + \frac{1}{y_j^p}$ we have that

$$\frac{1}{(k+1)^p} < \mu(y_j) = \frac{1}{y_j^p} \leq \frac{1}{k^p}.$$

Therefore

$$k^p \leq N_{k+1} - N_k \leq (k+1)^p.$$

Consider

$$\begin{aligned} B(t) = B(t_n) &= \sum_{j=0}^{n-1} Arctan\left(\frac{\beta\mu(t_j)}{t_j + \mu(t_j)\alpha}\right) \\ &= \sum_{j=0}^{n-1} Arctan\left(\frac{\beta}{t_j^{p+1} + \alpha}\right). \end{aligned}$$

To prove that (1) is oscillatory it suffices to show that

$$\sum_{j=0}^{\infty} Arctan\left(\frac{1}{t_j^{p+1}}\right) = \infty.$$

To show this note that

$$\begin{aligned} &\sum_{j=0}^{\infty} Arctan\left(\frac{1}{t_j^{p+1}}\right) \\ \geq\ &\sum_{j=0}^{\infty}\left(\frac{1}{t_j^{p+1}} - \frac{1}{3t_j^{3(p+1)}}\right) \\ =\ &\sum_{k=0}^{\infty}\sum_{j=N_k}^{N_{k+1}-N_k-1}\left(\frac{1}{t_j^{p+1}} - \frac{1}{3t_j^{3(p+1)}}\right) \\ \geq\ &\sum_{k=0}^{\infty}\left(\frac{k^p}{(k+1)^{p+1}} - \frac{(k+1)^p}{3k^{3(p+1)}}\right) \\ =\ &\infty. \end{aligned}$$

□

One might think that one could use the argument in the proof of Theorem 5 to show that if there is an increasing unbounded sequence of points $\{t_j\}$ in $\mathbb{T}$ with $\mu(t_j) = \frac{1}{t_j^p}$, then the Euler–Cauchy equation (1) is oscillatory on $\mathbb{T}$ in the complex characteristic roots case. The following example shows that the same type of argument does not work.

Example 1. *Assume that the Euler–Cauchy dynamic equation* (1) *has complex characteristic roots* $\alpha \pm i\beta$, $\beta > 0$ *and* $\mathbb{T} := \cup_{n=1}^{\infty}[(n-1)^2+1, n^2]$. *To see if* (1) *is oscillatory or not in this case we have by* (10)

$$\begin{aligned}
\lim_{t\to\infty} B(t) &= \int_1^{\infty} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau \\
= &\sum_{n=1}^{\infty}\int_{n^2}^{n^2+1} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau \\
+ &\sum_{n=1}^{\infty}\int_{n^2+1}^{(n+1)^2} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)d\tau. \qquad (14)
\end{aligned}$$

Consider the first term on the right hand side of equation (14). *This is the term we get by looking at the right scattered points in* $\mathbb{T}$. *Note that for* n_0 *sufficiently large*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_{n^2}^{n^2+1} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau \\
= &\sum_{n=1}^{\infty} Arctan\left(\frac{\beta\mu(n^2)}{n^2+\mu(n^2)\alpha}\right) \\
= &\sum_{n=1}^{\infty} Arctan\left(\frac{\beta}{n^2+\alpha}\right) \\
= &\sum_{n=1}^{n_0-1} Arctan\left(\frac{\beta}{n^2+\alpha}\right) + \sum_{n=n_0}^{\infty} Arctan\left(\frac{\beta}{n^2+\alpha}\right) \\
\le &\sum_{n=1}^{n_0-1} Arctan\left(\frac{\beta}{n^2+\alpha}\right) + \sum_{n=n_0}^{\infty} \left(\frac{\beta}{n^2+\alpha}\right) \\
< &\ \infty.
\end{aligned}$$

Hence unlike in the proof of Theorem 5 we do not get that this term corresponding to the right-scattered points is infinite. But now consider the second term on the *right hand side of equation* (14)

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \int_{n^2+1}^{(n+1)^2} \Im \left(\xi_{\mu(\tau)} \left(\frac{\alpha + i\beta}{\tau} \right) \right) d\tau \\
&= \sum_{n=1}^{\infty} \int_{n^2+1}^{(n+1)^2} \Im \left(\frac{\alpha + i\beta}{\tau} \right) d\tau \\
&= \sum_{n=1}^{\infty} \int_{n^2+1}^{(n+1)^2} \left(\frac{\beta}{\tau} \right) d\tau \\
&= \beta \sum_{n=1}^{\infty} \log \left(\frac{(n+1)^2}{n^2+1} \right) \\
&= \beta \log 2 + \beta \sum_{n=2}^{\infty} \log \left(\frac{(n+1)^2}{n^2+1} \right) \\
&= \beta \log 2 + \beta \sum_{n=2}^{\infty} \log \left(1 + \frac{2n}{n^2+1} \right) \\
&\geq \beta \log 2 + \beta \sum_{n=2}^{\infty} \left(\frac{2n}{n^2+1} - \frac{2n^2}{(n^2+1)^2} \right) \\
&= \infty.
\end{aligned}
$$

Hence we get that our equation is oscillatory.

Conjecture 1. *If the time scale* $\mathbb{T}$ *is an unbounded subset of the real interval* $(0, \infty)$ *and the Euler–Cauchy equation* (1) *has complex characteristic roots, then* (1) *is oscillatory on* $\mathbb{T}$.

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