A Nonstandard Fourier Inequality

Jonas Azzam, Bobbie Buchholz, Ian Grooms
Gretchen Hagge, Kyle Hays, Greg Norgard

Abstract

We consider a class of functions given by series of the form

$$h(x) := \sum_{k=1}^{\infty} \frac{f_k e^{a_k x} \sin(k^2 x)}{k^2},$$

where $f_k > 0$ for all k, $\lim f_k > 0$, and (a_k) is a bounded sequence. Such functions are continuous at all $x \in \mathbb{R}$, but not differentiable at 0. We prove that these functions satisfy $h(x) > mx^{1/2}$ for some $m > 0$ and all sufficiently small $x > 0$.

1. Introduction

Consider the function given by the Fourier Series:

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(k^2 x)}{k^2},$$

for $x \in \mathbb{R}$. Since the coefficients of $\sin(k^2 x)$ are summable, it is well-known that $f(x)$ converges for all x and is continuous for all x (see for instance Tolstov [2]). It is obvious that $f(0) = 0$. In this paper we study the behavior of $f(x)$, and functions like $f(x)$, for small $x > 0$.

It seems likely that $f(x) > 0$ for all sufficiently small $x > 0$, since every term is zero and increasing at $x = 0$. However, this is non-trivial to prove, since for each $x > 0$ there are infinitely many terms which are negative. One cannot analyze the sign of the derivative of f at $x = 0$, since $f'(0)$ does not exist; formal differentiation of $f(x)$ yields

$$f'(x) \text{``} = \text{''} \sum_{k=1}^{\infty} \cos(k^2 x),$$

which does not converge for any x.

We will show that for this $f(x)$, and similar functions, there exists $\beta > 0$ and $\gamma > 0$ such that

$$f(x) \geq \beta x^{1/2} \text{ for } x \in (0, \gamma]. \tag{1.1}$$

To illustrate this, we consider a graph of an approximation to $f(x)$ given by

$$g(x) = \sum_{k=1}^{10{,}000} \frac{\sin(k^2 x)}{k^2}.$$

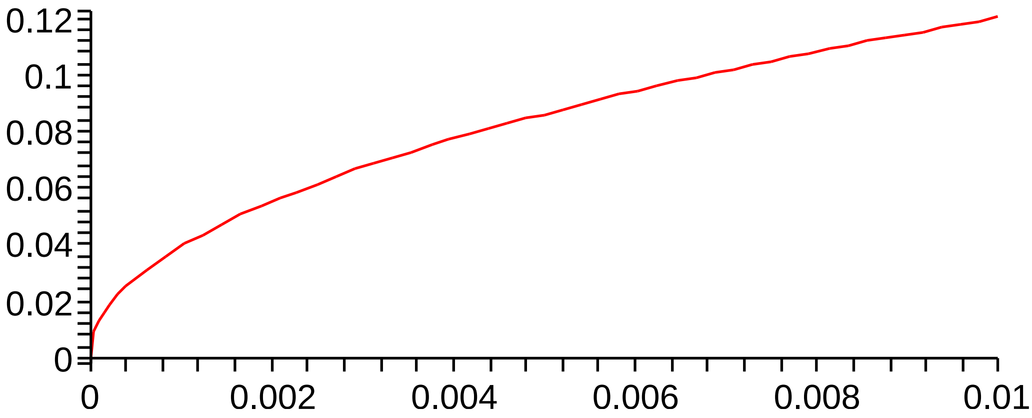

Figure 1.1: A graph of $g(x)$ near 0.

It appears from this graph that $g(x)$ approaches 0 almost vertically as $x \to 0$. The truncation g of f is a good approximation to $f(x)$, since

$$|g(x) - f(x)| = \left| \sum_{n=10{,}001}^{\infty} \frac{\sin(k^2 x)}{k^2} \right| \leq \sum_{n=10{,}001}^{\infty} \frac{1}{k^2} \leq \int_{10{,}000}^{\infty} \frac{1}{y^2}\, dy = \frac{1}{10,000}$$

for all $x \in \mathbb{R}$. However, g is very different from f in ways that are relevant to the behavior near zero. Every term of g is positive for all sufficiently small $x > 0$, and g is differentiable at $x = 0$. Therefore the graph of g can be approximated by the non-vertical line $y = 10,000x$ at 0. In particular, no truncation of f can satisfy condition (1.1).

The functions considered in this paper arise in the study of sampled-data control of infinite-dimensional systems, see for instance Logemann, *et al.* [1]. The need for the estimate obtained in this paper arose in the study of such systems. We were unable to find any results in the mathematical literature giving the result in Theorem 2.1 below.

2. Main Result

In this section we prove an inequality for a class of nonharmonic Fourier series.

Theorem 2.1 *Suppose* $f_k > 0$ *for* $k \in \mathbb{N}$, $\lim_{k\to\infty} f_k = f > 0$, *and* $(\alpha_k)_{k\in\mathbb{N}}$ *is a bounded sequence. Then there exists* $\beta, \gamma > 0$ *such that*

$$h(x) := \sum_{k=1}^{\infty} \frac{f_k e^{a_k x} \sin k^2 x}{k^2} \geq \beta x^{1/2} \text{ for } x \in (0, \gamma].$$

Proof: We first need to pick a constant θ such that

$$\theta^{1/2} \cos\theta - \frac{1}{\pi^{1/2}} > 0. \tag{2.1}$$

We can do this because $\varphi(\theta) := \theta^{1/2}\cos(\theta)$ is continuous on $[0, \pi/4]$ and

$$\varphi(0) = 0 < \pi^{-1/2} \text{ and } \varphi(\pi/4) = \pi^{1/2}/2^{3/2} > \pi^{-1/2}.$$

Since $\lim_{k\to\infty} f_k = f$, for every $\delta \in (0, f)$, there exists $K_\delta > 0$ so that

$$f - \delta < f_k < f + \delta \text{ for } k \geq K_\delta. \tag{2.2}$$

For each $\delta \in (0, f)$, let

$$x_\delta := \frac{\theta}{(K_\delta + 1)^2(1+\delta)}. \tag{2.3}$$

Hence

$$K_\delta + 1 < \left(\frac{\theta}{(1+\delta)x}\right)^{1/2} \text{ for } 0 < x < x_\delta.$$

Denoting the least integer of a real number a by $\lfloor a \rfloor$, we see that

$$K_\delta < \left\lfloor \left(\frac{\theta}{x_\delta(1+\delta)}\right)^{1/2} \right\rfloor \text{ for } 0 < x < x_\delta. \tag{2.4}$$

For each $\varepsilon > 0$, there exists x_ε such that

$$-\varepsilon < e^{a_k x} - 1 < \varepsilon \text{ for } 0 < x < x_\varepsilon, \tag{2.5}$$

since (a_k) is a bounded sequence.

Let $x^* = \min\{x_\delta, x_\varepsilon\}$ and consider $x \in (0, x^*)$. Let

$$N(x) := \left\lfloor \left(\frac{\theta}{x(1+\delta)} \right)^{1/2} \right\rfloor, \qquad N_\pi(x) := \left\lfloor \left(\frac{\pi}{x(1+\delta)} \right)^{1/2} \right\rfloor$$

We write

$$h(x) = h_+(x) + h_-(x),$$

where

$$h_+(x) = \sum_{k=1}^{N_\pi(x)} f_k \frac{e^{a_k x} \sin k^2 x}{k^2}$$

and

$$h_-(x) = \sum_{k=N_\pi(x)+1}^{\infty} f_k \frac{e^{a_k x} \sin k^2 x}{k^2}.$$

We will see that $h_+(x)$ is positive and satisfies the inequality that we want for $h(x)$, while $|h_-(x)|$ is small. Hence, our approach will be to get a lower bound on $h_+(x)$ and an upper bound on $|h_-(x)|$, and then use the reverse triangle inequality. We examine h_+ first. For $1 \le k \le N_\pi(x)$, it follows that

$$k^2 < (1+\delta)k^2 \le (1+\delta)N_\pi(x)^2 \le \frac{\pi}{x}. \tag{2.6}$$

Similarly, for $1 \le k \le N(x)$,

$$k^2 < (1+\delta)k^2 \le (1+\delta)N(x)^2 \le \frac{\theta}{x} \le \frac{\pi}{4x}. \tag{2.7}$$

¿From (2.6) we see that for $1 \le k \le N_\pi(x)$, $k^2 x \in (0, \pi)$, so

$$f_k e^{a_k x} \sin(k^2 x)/k^2 \ge 0 \text{ for } x \in (0, x^*),\ 1 \le k \le N_\pi(x). \tag{2.8}$$

Hence

$$h_+(x) = \Big(\sum_{k=1}^{K_\delta - 1} + \sum_{k=K_\delta}^{N(x)} + \sum_{k=N(x)}^{N_\pi(x)}\Big) f_k e^{a_k x} \frac{\sin(k^2 x)}{k^2} \ge \sum_{k=K_\delta}^{N(x)} f_k e^{a_k x} \frac{\sin(k^2 x)}{k^2}. \tag{2.9}$$

Using (2.5) and (2.2) we know that for $x \in (0, x^*)$,

$$\sum_{k=K_\delta}^{N(x)} f_k e^{a_k x} \frac{\sin(k^2 x)}{k^2} > \sum_{k=K_\delta}^{N(x)} (f - \delta)(1 - \varepsilon) \frac{\sin(k^2 x)}{k^2}. \tag{2.10}$$

Recall that

$$0 < \cos(t) \le \frac{\sin(t)}{t} \text{ for } t \in (0, \frac{\pi}{2}) \tag{2.11}$$

and that $\cos(\cdot)$ is a decreasing function on $(0, \pi/4)$. Hence, (2.11) and (2.7) yield

$$\cos(\theta) \le \cos(k^2 x) \le \frac{\sin(k^2 x)}{k^2 x} \text{ for } K_\delta \le k \le N(x).$$

It follows that

$$\sum_{k=K_\delta}^{N(x)} (f-\delta)(1-\varepsilon)\frac{\sin(k^2 x)}{k^2} \ge \sum_{k=K_\delta}^{N(x)} (f-\delta)(1-\varepsilon) x \cos(\theta).$$

$$= (f-\delta)(1-\varepsilon)\left(N(x) - K_\delta + 1\right) x \cos(\theta).$$

Combining this with (2.9) and (2.10) and using the fact that $\lfloor y \rfloor \ge y - 1$,

$$h_+(x) \ge (f-\delta)(1-\varepsilon)\left(\left(\frac{\theta}{(1+\delta)x}\right)^{1/2} - K_\delta\right) x \cos(\theta),$$

or

$$h_+(x) \ge x^{1/2}(f-\delta)(1-\varepsilon)\left(\left(\frac{\theta}{1+\delta}\right)^{1/2} - K_\delta x^{1/2}\right)\cos(\theta). \tag{2.12}$$

We now obtain an upper bound on $|h_-(x)|$.

$$|h_-(x)| = \left|\sum_{N_\pi(x)+1}^{\infty} f_k e^{a_k x} \frac{\sin(k^2 x)}{k^2}\right| \le \sum_{k=N_\pi(x)+1}^{\infty} \frac{f_k e^{a_k x}}{k^2}.$$

For $x \in (0, x^*)$, (2.2) and (2.5) apply, so

$$\sum_{k=N_\pi(x)+1}^{\infty} \frac{f_k e^{a_k x}}{k^2} \le \sum_{k=N_\pi(x)+1}^{\infty} \frac{(f+\delta)(1+\varepsilon)}{k^2}.$$

The integral test yields

$$\sum_{k=N_\pi(x)+1}^{\infty} \frac{1}{k^2} \le \int_{k=N_\pi(x)}^{\infty} \frac{1}{x^2} dx = \frac{1}{N_\pi(x)}.$$

Note that

$$N_\pi(x) > \left(\frac{\pi}{(1+\delta)x}\right)^{1/2} - 1,$$

so

$$|h_-(x)| \le (f+\delta)(1+\varepsilon)\frac{1}{\left[\left(\frac{\pi}{(1+\delta)x}\right)^{1/2} - 1\right]}.$$

or

$$|h_-(x)| \le x^{1/2}(f+\delta)(1+\varepsilon)\frac{1}{\left[\left(\frac{\pi}{(1+\delta)}\right)^{1/2} - x^{1/2}\right]}. \tag{2.13}$$

We are now ready to combine our lower bound on $h_+(x)$ and our upper bound on $|h_-(x)|$ for $x \in (0, x^*)$. Define

$$b(x,\delta,\varepsilon) := \left[(f-\delta)(1-\varepsilon)\left(\frac{\theta}{1+\delta}\right)^{1/2}\cos\theta - \frac{(f+\delta)(1+\varepsilon)}{\left[\left(\frac{\pi}{1+\delta}\right)^{1/2} - x^{1/2}\right]}\right].$$

Then (2.12) and (2.13) yield

$$h(x) \ge x^{1/2}\left[b(x,\delta,\varepsilon) - K_\delta(f-\delta)(1-\varepsilon)x^{1/2}\cos(\theta)\right]. \tag{2.14}$$

¿From (2.1),

$$b(0,0,0) = f\left[\theta^{1/2}\cos\theta - \frac{1}{\pi^{1/2}}\right] > 0. \tag{2.15}$$

It is easy to see that b is continuous in $\{(x,\delta,\varepsilon) \mid x \ge 0, \delta \ge -1, \varepsilon \in \mathbb{R}\}$. Hence there exists $\varepsilon \in (0,1)$, $\delta \in (0,f)$, $\eta > 0$ and $\beta > 0$ such that

$$b(x,\delta,\varepsilon) > 2\beta \text{ for } x \in (0,\eta). \tag{2.16}$$

Note that this choice of ε and δ fixes x^* and also fixes K_δ in (2.14). Hence we see that there exists $\gamma \in (0, \min\{\eta, x^*\}]$ such that

$$b(x,\delta,\varepsilon) - K_\delta(f-\delta)(1-\varepsilon)x^{1/2}\cos(\theta) > \beta \text{ for } x \in (0,\gamma).$$

Combining this with (2.14),

$$h(x) > \beta x^{1/2} \text{ for } x \in (0,\gamma),$$

completing the proof. □

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References

[1] H. Logemann, R. Rebarber and S. Townley. Stability of infinite-dimensional sampled-data systems, *Trans. Amer. Math. Soc.*, **355** (2003), 3301–3328.

[2] G. Tolstov, *Fourier Series*, Dover Publications, New York, 1962.