First-Order Recurrence Relations on Isolated Time Scales

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1. Some Preliminaries on time scales

A recent cover story in "New Scientist," [3] reports of many important applications concerning dynamic equations on time scales. Some of these applications include a model of the West Nile virus, of electrical activity in the heart, of the stock market, of combustion in engines, of bulimia and of population models that vary in continuous time and discrete time, as well as the study of the interfaces of nanoscale structures embedded in other materials.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Examples include $\mathbb{R}$, the integers $\mathbb{Z}$, the harmonic numbers $\{H_n\}_0^\infty$ where $H_0 = 0$ and $H_n = \sum_{i=1}^n \frac{1}{i}$, and $\overline{q^{\mathbb{Z}}} = \{0\} \cup \{q^n | \, n \in \mathbb{Z}\}$ where $q > 1$. On any time scale $\mathbb{T}$ we define the forward and backward jump operators by:

$$\sigma(t) := \inf\{s \in \mathbb{T} | \; s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} | \; s < t\}, \tag{1.1}$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A time scale $\mathbb{T}$ is said to be an isolated time scale provided given any $t \in \mathbb{T}$, there is a $\delta > 0$ such that $(t-\delta, t+\delta) \cap \mathbb{T} = \{t\}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) \equiv 0$$

[1]This work was done during the summer of 2003 at the REU site University of Nebraska-Lincoln, NSF grant number 25-0517-0041-001, under the guidance of Dr. Allan Peterson.

for $\mathbb{T}=\mathbb{Z}$ we have

$$\sigma(t)=t+1,\quad \rho(t)=t-1,\quad \mu(t)\equiv 1,$$

for the case $q^{\mathbb{N}}=\{q^n|\, n\in\mathbb{N}\}$ we have

$$\sigma(t)=qt,\quad \rho(t)=\frac{t}{q},\quad \mu(t)=(q-1)t,$$

and for the time scale $\mathbb{N}_0^b:=\{n^b|\, n\in\mathbb{N}_0\}$ where $b\in\mathbb{N}$,

$$\sigma(t)=(t^{\frac{1}{b}}+1)^b,\quad \rho(t)=(t^{\frac{1}{b}}-1)^b,\quad \mu(t)=\sum_{k=0}^{b-1}\binom{b}{k}t^{\frac{k}{b}}.$$

For a function $f:\mathbb{T}\to\mathbb{R}$ the (delta) derivative $f^{\Delta}(t)$ at $t\in\mathbb{T}^{\kappa}$ is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property such that given any $\epsilon>0$, there exists a neighborhood U of t such that

$$\big|\,[f(\sigma(t))-f(s)]-f^{\Delta}(t)(\sigma(t)-s)\big|\le\epsilon|\sigma(t)-s|\quad\text{for all}\quad s\in\mathbb{T}\cap U.$$

The delta derivative is given by

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)} \tag{1.2}$$

if f is continuous at t and t is right-scattered. If t is right-dense then the derivative is given by

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{t-s}=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}, \tag{1.3}$$

provided this limit exists. This definition can be generalized to the case where the range of f is any Banach space. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd-continuous) if it is continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. A useful formula is

$$f^{\sigma}:=f\circ\sigma=f+\mu f^{\Delta}. \tag{1.4}$$

For $a,b\in\mathbb{T}$, and a differentiable function F, the Cauchy integral of $F^{\Delta}=f$ is defined by

$$\int_a^b f(t)\Delta t=\int_a^b F^{\Delta}(t)\Delta t=F(b)-F(a).$$

We shall make particular use of the formula

$$\int_a^b f(t)\Delta t=\sum_{s\in[a,b)} f(s)\mu(s),\quad a<b \tag{1.5}$$

(see [1]) which holds for isolated time scales.

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}.$$

It turns out that the set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group (see Theorem 2.7 and Exercise 2.26 in [1]) under the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu pq$$

and the inverse $\ominus p$ of the function p is defined by

$$\ominus p := \frac{-p}{1 + \mu p}.$$

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are rd–continuous and regressive by $\mathcal{R}$. If $p \in \mathcal{R}$, then we can define the exponential function by

$$e_p(t, s) := \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

where Log denotes the principal logarithm function. Alternately, for $p \in \mathcal{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP

$$x^\Delta = p(t)x, \quad x(t_0) = 1.$$

We define

$$\mathcal{R}^+ := \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0, t \in \mathbb{T}\}.$$

We shall be making heavy use of the properties

(1) $e_0(t, s) \equiv 1, e_p(t, t) \equiv 1$
(2) $e_p(\sigma(t), t_0) = [1 + \mu(t)p(t)]e_p(t, t_0)$
(3) $\frac{1}{e_p(t,t_0)} = e_{\ominus p}(t, t_0) = e_p(t_0, t)$
(4) $e_p(t, t_0)e_q(t, t_0) = e_{p\oplus q}(t, t_0)$
(5) $e_p(t, r)e_p(r, s) = e_p(t, s)$

where $p, q \in \mathcal{R}$ (see Bohner and Peterson [1]).

Also if $p \in \mathcal{R}$, then $e_p(t, s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathcal{R}^+$, then $e_p(t, t_0)$ is always positive. For $\alpha \in \mathbb{R}$, on the respective time scales, the exponential $e_\alpha(t, t_0)$ is

$\mathbb{R}$	:	$e^{\alpha(t-t_0)}$
$h\mathbb{Z}$	:	$(1 + \alpha h)^{\frac{t-t_0}{h}}$
$\overline{q^{\mathbb{Z}}}$	:	$\prod_{s\in[t_0,t)}[1 + (q - 1)\alpha s], t > t_0$
$\{H_n\}_0^\infty$	:	$\binom{n+\alpha-t_0}{n-t_0}$, $t = \sum_{k=1}^n \frac{1}{k}$.

2. General Results

We will be principally concerned with the dynamic equation

$$y^\sigma - p(t)y = r(t), \quad t \in \mathbb{T}^\kappa \tag{2.1}$$

on isolated time scales, where $p(t) \neq 0$, $\forall t \in \mathbb{T}^\kappa$. In particular, solutions of the corresponding homogeneous problem $u^\sigma - p(t)u = 0$ can be used to find explicit forms for generalized exponential functions on a particular time scale.

We begin by finding a variation of constants formula for (2.1).

Theorem 2.1. ***Variation of Constants for First Order Recurrence Relations*** *Assume* $p(t) \neq 0$, $\forall t \in \mathbb{T}^\kappa$. *Then the unique solution to the IVP*

$$y^\sigma - p(t)y = r(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_{\frac{p-1}{\mu}}(t, t_0)y_0 + \int_{t_0}^{t} e_{\frac{p-1}{\mu}}(t, \sigma(s))\frac{r(s)}{\mu(s)}\Delta s.$$

We shall now give two proofs, one approaching the problem as a first order dynamic equation, and a second following a method used in the study of difference equations.

Proof. Using formula (1.4), we may rewrite the corresponding homogeneous equation as follows:

$$\begin{aligned} u^\sigma &= p(t)u \\ u + \mu(t)u^\Delta &= p(t)u \\ \mu(t)u^\Delta &= (p(t) - 1)u \\ u^\Delta &= \frac{p(t) - 1}{\mu(t)}u \end{aligned}$$

and so by the definition of the generalized exponential as the solution of an IVP, we get that

$$u(t) = e_{\frac{p-1}{\mu}}(t, t_0)u_0 \tag{2.2}$$

where $u_0 = u(t_0)$. Using the variation of constants formula found in Bohner and Peterson [1], we find the general solution to equation (2.1) is

$$y(t) = e_{\frac{p-1}{\mu}}(t, t_0)y_0 + \int_{t_0}^{t} e_{\frac{p-1}{\mu}}(t, \sigma(s))\frac{r(s)}{\mu(s)}\Delta s \tag{2.3}$$

where the arguments have been supressed in the subscripts, or alternately

$$y(t) = e_{\frac{p-1}{\mu}}(t, t_0)\left(y_0 + \int_{t_0}^{t} \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s, t_0)}\Delta s\right). \tag{2.4}$$

We now derive the variation of constants formula using a method analogous to that used in the difference equations case. First we prove a lemma.

Lemma 2.1. *The exponential function* $e_{\frac{p-1}{\mu}}(t,t_0)$ *is given by*

$$e_{\frac{p-1}{\mu}}(t,t_0) = \begin{cases} \prod_{\tau\in[t_0,\,t)} p(\tau), & t \geq t_0 \\ \prod_{\tau\in[t,\,t_0)} \frac{1}{p(\tau)}, & t < t_0 \end{cases} .$$

Proof. Since the exponential $e_{\frac{p-1}{\mu}}(t,t_0)$ is the unique solution to $u^\sigma = p(t)u$ with $u(t_0) = 1$, we note that $\prod_{\tau\in[t_0,\,t_0)} p(\tau) = 1$ using the convention that an empty product is the identity, and for $t > t_0$ one may simply iterate the formula $u^\sigma = p(t)u$. The case $t < t_0$ is similar. □

Consider the dynamic equation $y^\sigma - p(t)y = r(t)$ where $p, r \in C_{rd}$ and $p(t) \neq 0$, $\forall t \in \mathbb{T}$, and let u be a nonzero solution of the corresponding homogeneous equation $u^\sigma = p(t)u$ which by Lemma 2.1 is $\prod_{\tau\in[t_0,\,t)} p(\tau)$. Let us assume that y is a solution to (2.1) and now divide (2.1) by $u^\sigma(t)\mu(t)$ to get

$$\frac{y^\sigma(t)}{u^\sigma(t)\mu(t)} - \frac{p(t)y(t)}{u^\sigma(t)\mu(t)} = \frac{r(t)}{u^\sigma(t)\mu(t)}.$$

Hence

$$\frac{y^\sigma(t)}{u^\sigma(t)\mu(t)} - \frac{p(t)y(t)}{u^\sigma(t)\mu(t)} = \frac{\frac{y^\sigma(t)}{u^\sigma(t)} - \frac{y(t)}{u(t)}}{\mu(t)} = \left(\frac{y(t)}{u(t)}\right)^\Delta = \frac{r(t)}{u^\sigma(t)\mu(t)}.$$

Now, we may integrate both sides from t_0 to t to find that

$$\frac{y(t)}{u(t)} - \frac{y_0}{u_0} = \int_{t_0}^{t} \frac{r(s)}{u^\sigma(s)\mu(s)}\Delta s$$

and thus we have

$$y(t) = u(t)\left(y_0 + \int_{t_0}^{t} \frac{r(s)}{u^\sigma(s)\mu(s)}\right)\Delta s$$

which, upon noting $u(t) = e_{\frac{p-1}{\mu}}(t,t_0)u_0$, is exactly (2.3). □

See Examples 3.1, 3.2, 3.4, and 3.7 for simple applications of the variation of constants formula.

For the case $\mathbb{T} = \mathbb{Z}$, we know in the study of difference equations (see [2]) that the solution to the problem $y(t+1) - p(t)y(t) = r(t)$ is

$$u(t)\left(y_0 + \sum \frac{r(t)}{u(t+1)}\right) \tag{2.5}$$

where $u(t)$ is a nonzero solution to $u(t+1) = p(t)u(t)$. However, by the definition of the generalized exponential, this function $u(t)$ is a constant multiple of the exponential $e_{\frac{p-1}{\mu}}(t,t_0)$, and $p(t)e_{\frac{p-1}{\mu}}(t,t_0) = e^\sigma_{\frac{p-1}{\mu}}(t,t_0)$. Since on $\mathbb{Z}$ we have $\mu \equiv 1$, the Cauchy integral is exactly an indefinite sum, and thus

$$u(t)\left(y_0 + \sum \frac{r(t)}{u(t+1)}\right) = e_{\frac{p-1}{\mu}}(t,t_0)\left(y_0 + \sum \frac{r(s)}{e^{\sigma}_{\frac{p-1}{\mu}}(s,t_0)}\right)$$

$$= e_{\frac{p-1}{\mu}}(t,t_0)\left(y_0 + \int \frac{r(s)}{\mu(s)e^{\sigma}_{\frac{p-1}{\mu}}(s,t_0)}\Delta s\right)$$

$$= e_{\frac{p-1}{\mu}}(t,t_0)\left(y_0 + \int \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s,t_0)}\Delta s\right).$$

Theorem 2.2. ***Factorization*** *A solution to the dynamic equation* $u^\sigma = p(t)u$, *where* $p(t) = p_1(t)p_2(t)\dots p_n(t)$ *is*

$$u(t) = e_{\frac{p_1p_2\cdots p_n-1}{\mu}}(t,t_0) = e_{\frac{p_1-1}{\mu}}(t,t_0)e_{\frac{p_2-1}{\mu}}(t,t_0)\dots e_{\frac{p_n-1}{\mu}}(t,t_0).$$

Proof. We note that

$$\frac{a-1}{\mu} \oplus \frac{b-1}{\mu} = \frac{ab-1}{\mu}$$

for any a, b, and by induction that

$$e_{\frac{p_1p_2\cdots p_n-1}{\mu}}(t,t_0) = e_{\frac{p_1-1}{\mu}\oplus\frac{p_2-1}{\mu}\oplus\dots\oplus\frac{p_n-1}{\mu}}(t,t_0)$$

$$= e_{\frac{p_1-1}{\mu}}(t,t_0)e_{\frac{p_2-1}{\mu}}(t,t_0)\dots e_{\frac{p_n-1}{\mu}}(t,t_0).$$

□

A basic application is given in Example 3.3.

2.1. **Series Solution.** A useful tool for looking at isolated time scales is the counting function n_t defined

$$n_t(t,s) := \int_s^t \frac{\Delta\tau}{\mu(\tau)}.$$

Note that using (1.5) the values of this function are integers, where the value of the function n_t counts the number of points in the half open interval $[s,t)$ for $t \geq s$ and is the negative of the number of points in $[t,s)$ for $t < s$. On an isolated time scale, $n_t(\sigma^k(s),s) = k$, with the conventions $\sigma^{-k}(t) = \rho^k(t)$ and $\sigma^0(t) = t$. For example, for the time scale $\mathbb{N}^b, b > 0$,

$$n_t(t,s) = \sqrt[b]{t} - \sqrt[b]{s}$$

and for $\overline{q^{\mathbb{Z}}}$ the counting function is given by

$$n_t(t,s) = \frac{\ln t - \ln s}{\ln q} = \log_q\left(\frac{t}{s}\right).$$

The counting function provides an enumeration of the time scale which can be quite useful in formulas for exponential functions, particularly when considering the product formula for exponentials given in Lemma 2.1. We use this counting function in the following theorem.

Theorem 2.3. ***Series Solution*** *The sum*

$$\frac{y(\sup\mathbb{T})}{p(t)p^{\sigma}(t)p^{\sigma^2}(t)\dots p(\sup\mathbb{T}^{\kappa})}+\sum_{k=0}^{n_t(\sup\mathbb{T}^{\kappa},t)}\frac{-r^{\sigma^k}(t)}{p(t)p^{\sigma}(t)p^{\sigma^2}(t)\dots p^{\sigma^k}(t)} \tag{2.6}$$

is a solution to (2.1) *where* $\sup\mathbb{T}=\max\mathbb{T}\in\mathbb{R}$, *and if* $\sup\mathbb{T}=\infty$, *then a solution is given by the infinite series*

$$y(t)=\sum_{k=0}^{\infty}\frac{-r^{\sigma^k}(t)}{p(t)p^{\sigma}(t)p^{\sigma^2}(t)\dots p^{\sigma^k}(t)} \tag{2.7}$$

whenever this series converges, where y^{σ^k} *denotes* $y(\overbrace{\sigma(\sigma(\dots(\sigma(}^{k\ times}t))\dots)))$.

Proof. Observe that equation $y^{\sigma}-p(t)y=r(t)$ can be re-expressed in the form $y=\frac{-r(t)+y^{\sigma}}{p(t)}$, and that $y^{\sigma}=\frac{-r^{\sigma}(t)+y^{\sigma\sigma}}{p^{\sigma}(t)}$, and thus $y=\frac{-r+\frac{-r^{\sigma}+y^{\sigma\sigma}}{p^{\sigma}}}{p}$. Continuing this process,

$$y(t)=\frac{-r(t)+\frac{-r^{\sigma}(t)+\frac{-r^{\sigma^2}(t)+\frac{\dots+y(\sup\mathbb{T})}{\dots}}{p^{\sigma^2}(t)}}{p^{\sigma}(t)}}{p(t)},$$

when $\sup\mathbb{T}$ exists, and formally we get

$$y(t)=\frac{-r(t)+\frac{-r^{\sigma}(t)+\frac{-r^{\sigma^2}(t)+\frac{-r^{\sigma^3}(t)+\dots}{p^{\sigma^3}(t)}}{p^{\sigma^2}(t)}}{p^{\sigma}(t)}}{p(t)}$$

if $\sup\mathbb{T}=\infty$.

Consider first the case where $\sup\mathbb{T}$ is finite. The ascending fraction can be expanded into the sum

$$\frac{-r(t)}{p(t)}+\frac{-r^{\sigma}(t)}{p(t)p^{\sigma}(t)}+\dots+\frac{-r(\sup\mathbb{T}^{\kappa})}{p(t)p^{\sigma}(t)\dots p(\sup\mathbb{T}^{\kappa})}+\frac{y(\sup\mathbb{T})}{p(t)p^{\sigma}(t)\dots p(\sup\mathbb{T}^{\kappa})}.$$

Since there is one summand with the function $r(t)$ for each point in the interval $[t,\sup\mathbb{T})$, the number of such summands is $n_t(\sup\mathbb{T},t)$ and thus starting our index at zero, these terms combine as

$$\sum_{k=0}^{n_t(\sup\mathbb{T},t)-1}\frac{-r^{\sigma^k}(t)}{p(t)p^{\sigma}(t)p^{\sigma^2}(t)\dots p^{\sigma^k}(t)}=\sum_{k=0}^{n_t(\sup\mathbb{T}^{\kappa},t)}\frac{-r^{\sigma^k}(t)}{p(t)p^{\sigma}(t)p^{\sigma^2}(t)\dots p^{\sigma^k}(t)}.$$

For $\sup \mathbb{T} = \infty$, formally expanding the ascending continued fraction yields the infinite series

$$\sum_{k=0}^{\infty} \frac{-r^{\sigma^k}}{pp^{\sigma}p^{\sigma^2}p^{\sigma^3}\dots p^{\sigma^k}}.$$

Assume that this infinite series converges for all $t \in \mathbb{T}$ and denote it by $w(t)$. Then

$$\begin{aligned} w(t) &= \sum_{k=0}^{\infty} \frac{-r^{\sigma^k}(t)}{p(t)p^{\sigma}(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^k}(t)} \\ p(t)w(t) &= -r(t) + \sum_{k=1}^{\infty} \frac{-r^{\sigma^k}(t)}{p^{\sigma}(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^k}(t)} \\ p(t)w(t) + r(t) &= \sum_{k=1}^{\infty} \frac{-r^{\sigma^k}(t)}{p^{\sigma}(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^k}(t)}. \end{aligned}$$

Hence we see that

$$\begin{aligned} p(t)w(t) + r(t) &= \sum_{k=1}^{\infty} \frac{-r^{\sigma^k}(t)}{p^{\sigma}(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^k}(t)} \\ &= \sum_{k=0}^{\infty} \frac{-r^{\sigma^{k+1}}(t)}{p^{\sigma}(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^{k+1}}(t)} \\ &= w^{\sigma}(t) \end{aligned}$$

and $w(t)$ is a solution to (2.1). $\square$

We now derive some alternate formulas for the solutions given in Theorem 2.3. We note that

$$\prod_{k=0}^{m} p^{\sigma^k}(t) = e_{\frac{p-1}{\mu}}(\sigma^{m+1}(t), t) = p^{\sigma^m}(t)e_{\frac{p-1}{\mu}}(\sigma^m(t), t)$$

by Lemma 2.1, and thus along with (1.5), we see that

$$\sum_{k=0}^{n_t(t^*,t)} \frac{-r^{\sigma^k}(t)}{p^{\sigma}(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^k}(t)} = -\int_t^{\sigma(t^*)} \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s,t)}\Delta s.$$

We may therefore re-express the summations in our theorem with the integrals

$$-e_{\frac{p-1}{\mu}}(t,t_0)\int_t^{\sup \mathbb{T}} \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s,t_0)}\Delta s$$

for $\sup \mathbb{T}$ finite, and

$$-e_{\frac{p-1}{\mu}}(t,t_0)\int_t^\infty \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s,t_0)}\Delta s$$

for $\sup \mathbb{T} = \infty$. Thus we find the initial condition must be

$$w_0 = -\int_{t_0}^{\sup \mathbb{T}} \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s,t_0)}\Delta s + \frac{y(\sup \mathbb{T})}{e_{\frac{p-1}{\mu}}(\sup \mathbb{T},t_0)}$$

when $\sup \mathbb{T}$ exists, else

$$w_0 = \sum_{k=0}^{\infty} \frac{-r^{\sigma^k}(t_0)}{p(t_0)p^{\sigma}(t_0)p^{\sigma^2}(t_0)\dots p^{\sigma^k}(t_0)} = -\int_{t_0}^{\infty} \frac{r(s)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s,t_0)}\Delta s.$$

On the time scale $\mathbb{Z}$ with $p(t) = t$ this leads to a factorial series (see Example 3.4 in [2]). It is also of interest to consider this series solution on $\mathbb{T} = \mathbb{R}$, where all points are right-dense. Thus the equation $y^\sigma - p(t)y = r(t)$ becomes simply $y - p(t)y = r(t)$ and $y = \frac{r(t)}{1-p(t)}$. The series is then

$$y(t) = \sum_{k=0}^{\infty} \frac{-r(t)}{p^k(t)} = -r(t)\sum_{k=0}^{\infty} \frac{1}{p^k(t)} = \frac{r(t)}{1-p(t)}$$

which is a geometric series which converges for $|p(t)| > 1$, exactly as we expected.

3. Examples

In this section we give several examples following from our previous results. The first two are direct applications of the variation of constants formula.

Example 3.1. Solve the dynamic equation

$$y^\sigma - qy = q - 1 \tag{3.1}$$

on the time scale $\mathbb{T} = q^{\mathbb{N}}$. In this case $p(t) = q$, $r(t) = q-1$, and $\mu(t) = t(q-1)$. By the variation of constants formula given in Theorem 2.1, we obtain

$$\begin{aligned} y(t) &= e_{\frac{(q-1)}{t(q-1)}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{(q-1)}{t(q-1)}}(t,\sigma(\tau))\frac{(q-1)}{\tau(q-1)}\Delta\tau \\ &= e_{\frac{1}{t}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{1}{t}}(t,t_0)e_{\frac{1}{t}}(t_0,\sigma(\tau))\frac{1}{\tau}\Delta\tau \\ &= e_{\frac{1}{t}}(t,t_0)y_0 + e_{\frac{1}{t}}(t,t_0)\int_{t_0}^{t} e_{\frac{1}{t}}(t_0,\sigma(\tau))\frac{1}{\tau}\Delta\tau \\ &= e_{\frac{1}{t}}(t,t_0)y_0 + e_{\frac{1}{t}}(t,t_0)\int_{t_0}^{t} \frac{1}{qe_{\frac{1}{t}}(\tau,t_0)}\frac{1}{\tau}\Delta\tau. \end{aligned}$$

Note that

$$\ominus\frac{1}{t} = \frac{\frac{-1}{t}}{1+\frac{1}{t}(tq-t)} = \frac{\frac{-1}{t}}{1+q-1} = \frac{-1}{qt}.$$

Therefore

$$\begin{aligned} \int_{t_0}^{t} \frac{1}{qe_{\frac{1}{t}}(\tau,t_0)}\frac{1}{\tau}\Delta\tau &= \int_{t_0}^{t} \frac{1}{q\tau}e_{\ominus\frac{1}{t}}(\tau,t_0)\Delta\tau \\ &= [-e_{\ominus\frac{1}{t}}(\tau,t_0)]_{t_0}^{t} \\ &= [-e_{\ominus\frac{1}{t}}(t,t_0)+1], \end{aligned}$$

and we get that

$$\begin{aligned} y(t) &= e_{\frac{1}{t}}(t,t_0)y_0 + e_{\frac{1}{t}}(t,t_0)[-e_{\ominus\frac{1}{t}}(t,t_0)+1] \\ &= e_{\frac{1}{t}}(t,t_0)y_0 - 1 + e_{\frac{1}{t}}(t,t_0) \\ &= (y_0+1)e_{\frac{1}{t}}(t,t_0) - 1. \end{aligned}$$

By Lemma 2.1, for $t \geq t_0$

$$e_{\frac{1}{t}}(t,t_0) = \prod_{\tau\in[t_0,t)} q = \frac{q^n}{q^{n_0}} = \frac{t}{t_0}.$$

So $y(t) = Ct - 1$, where $C = \frac{y_0+1}{t_0}$, is a general solution to (3.1).

Example 3.2. Consider the dynamic equation

$$y^\sigma - \frac{\mu(t)}{\mu(t+1)}y = 1 \tag{3.2}$$

on the so-called time scale $\mathbb{T} = \{H_n\}_{n=0}^{\infty}$ of Harmonic numbers [1]. In this case $p(t) = \frac{\mu(t)}{\mu(t+1)}$ and $r(t) = 1$. By the variation of constants formula

(Theorem 2.1), we obtain for $t \geq t_0$

$$\begin{aligned}
y(t) &= e_{\frac{\frac{\mu(t)}{\mu(t+1)}-1}{\mu(t)}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{\frac{\mu(t)}{\mu(t+1)}-1}{\mu(t)}}(t,\sigma(\tau))\frac{1}{\mu(\tau)}\Delta\tau \\
&= e_{\frac{\mu(t)-\mu(t+1)}{\mu(t)\mu(t+1)}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{\mu(t)-\mu(t+1)}{\mu(t)\mu(t+1)}}(t,\sigma(\tau))\frac{1}{\mu(\tau)}\Delta\tau \\
&= e_{\frac{1}{\mu(t+1)}-\frac{1}{\mu(t)}}(t,t_0)y_0 + \int_{t_0}^{t} e_{\frac{1}{\mu(t+1)}-\frac{1}{\mu(t)}}(t,\sigma(\tau))\frac{1}{\mu(\tau)}\Delta\tau.
\end{aligned}$$

But $\mu(t) = \frac{1}{n+1}$, so

$$\begin{aligned}
y(t) &= e_{(n+2)-(n+1)}(t,t_0)y_0 + \int_{t_0}^{t} e_{(n+2)-(n+1)}(t,\sigma(\tau))\frac{1}{\mu(\tau)}\Delta\tau \\
&= e_1(t,t_0)y_0 + \int_{t_0}^{t} e_1(t,\sigma(\tau))\frac{1}{\mu(\tau)}\Delta\tau \\
&= e_1(t,t_0)y_0 + \sum_{s\in[t_0,t)} \mu(s)e_1(t,\sigma(s))\frac{1}{\mu(s)} \\
&= e_1(t,t_0)y_0 + \sum_{s\in[t_0,t)} e_1(t,\sigma(s)) \\
&= e_1(t,t_0)y_0 + \sum_{s\in[t_0,t)} e_1(t,t_0)e_1(t_0,\sigma(s)) \\
&= e_1(t,t_0)y_0 + e_1(t,t_0)\sum_{s\in[t_0,t)} \frac{1}{e_1(\sigma(s),t_0)} \\
&= e_1(t,t_0)\left(y_0 + \sum_{s\in[t_0,t)} \frac{1}{e_1(\sigma(s),t_0)}\right).
\end{aligned}$$

If $t_0 = 0$, then we can use the fact (see page 74 in [1]) that $e_\alpha(t,t_0) = \binom{n+\alpha-t_0}{n-t_0}$ where $t = \sum_{i=1}^{n} \frac{1}{i}$ so that

$$\begin{aligned} y(t) &= \binom{n+1-0}{n-0}\left(y_0 + \sum_{s\in[t_0,t)} \frac{1}{\binom{n+2-0}{n+1-0}}\right) \\ &= (n+1)\left(y_0 + \sum_{s\in[t_0,t)} \frac{1}{n+2}\right) \\ &= (n+1)\left(y_0 + \sum_{k=0}^{n_t} \frac{1}{k+2}\right) \\ &= (n+1)\left(y_0 + \sum_{k=1}^{n_t} \frac{1}{k+1}\right) \\ &= (n+1)\left(y_0 + \sigma(t) - 1\right). \end{aligned}$$

Hence a general solution of equation (3.2) is $y(t) = (n+1)\left(y_0 + \sigma(t) - 1\right)$.

Example 3.3. For an example of Theorem 2.2, consider the homogeneous dynamic equation

$$u^\sigma - (qt^2 - q)u = 0 \tag{3.3}$$

on the time scale $\mathbb{T} = q^{\mathbb{N}_0}$. In this case $p(t) = (qt^2 - q)$ and $\mu(t) = (q-1)t$. But the original equation can be written in the form

$$u^\sigma - q(t^2 - 1)u = 0.$$

Hence, we can apply Theorem 2.2. Thus it is sufficient to solve for $u_1(t)$ where $p_1(t) = q$ and $u_2(t)$ where $p_2(t) = t^2 - 1$. So first consider $u_1^\sigma - qu_1 = 0$. Using equation (2.2), we obtain

$$u_1(t) = e_{\frac{q-1}{\mu(t)}}(t,t_0)u_0 = e_{\frac{q-1}{(q-1)t}}(t,t_0)u_0 = e_{\frac{1}{t}}(t,t_0)u_0.$$

By Example 3.1, $u_1(t) = u_0 \frac{t}{t_0}$.

Next consider $u_2^\sigma - (t^2-1)u_2 = 0$. As before, by equation (2.2) we obtain

$$u_2(t) = e_{\frac{(t^2-1)-1}{\mu(t)}}(t,t_0)u_0 = e_{\frac{t^2-2}{(q-1)t}}(t,t_0)u_0.$$

But by Lemma 2.1,

$$u_2(t) = e_{\frac{t^2-2}{(q-1)t}}(t,t_0)u_0 = u_0 \prod_{\tau\in[t_0,t)} (\tau^2 - 1).$$

Thus, by Theorem 2.2, we can write the general solution of equation (3.3) as

$$u(t) = u_0 e_{\frac{qt^2-q-1}{(q-1)t}}(t, t_0) = u_0 \frac{t}{t_0} \prod_{\tau \in [t_0, t)} (\tau^2 - 1).$$

Example 3.4. By solving initial value problems, in some cases we are able to find formulas for exponential functions. Consider the nonhomogeneous IVP

$$y^\sigma - qy = q^{t+1}, \quad y(0) = 0 \tag{3.4}$$

on the time scale $\mathbb{T} = \mathbb{Z}$. In this case $p(t) = q$, $r(t) = q^{t+1}$ and $\mu(t) = 1$. From the variation of constants formula equation (2.2), we obtain

$$\begin{aligned} y(t) &= \int_0^t e_{q-1}(t, \sigma(\tau)) q^{\tau+1} \Delta\tau \\ &= e_{q-1}(t, 0) \int_0^t \frac{1}{e_{q-1}(\sigma(\tau), 0)} q^{\tau+1} \Delta\tau \\ &= e_{q-1}(t, 0) \int_0^t \frac{1}{q e_{q-1}(\tau, 0)} q^{\tau+1} \Delta\tau \\ &= e_{q-1}(t, 0) \int_0^t \frac{1}{q} e_{\ominus(q-1)}(\tau, 0) q^{\tau+1} \Delta\tau. \end{aligned}$$

Note that

$$\ominus(q-1) = \frac{-(q-1)}{1+q-1} = \frac{1-q}{q}$$

and

$$e_{\frac{1-q}{q}}(t, 0) = \left(1 + \frac{1-q}{q}\right)^t = \left(\frac{1}{q}\right)^t.$$

Hence,

$$\begin{aligned} y(t) &= e_{q-1}(t, 0) \int_0^t \frac{1}{q} \left(\frac{1}{q}\right)^\tau q^{\tau+1} \Delta\tau \\ &= e_{q-1}(t, 0) \int_0^t \left(\frac{1}{q}\right)^{\tau+1} q^{\tau+1} \Delta\tau \\ &= e_{q-1}(t, 0) \int_0^t 1 \Delta\tau \\ &= t e_{q-1}(t, 0) \\ &= t(1+q-1)^t \\ &= tq^t \end{aligned}$$

is the solution to the IVP (3.4).

In the next two examples we consider homogeneous IVP's in order to find a formula for the respective exponential functions.

Example 3.5. Consider the IVP

$$u^\sigma - \frac{1+\sqrt{t}}{\sqrt{t}}u = 0, \quad u(1) = 1 \tag{3.5}$$

on the time scale $\mathbb{T} = \mathbb{N}^2 := \{t |\ t = n^2, n \in \mathbb{N}\}$. In this case $p(t) = \frac{1+\sqrt{t}}{\sqrt{t}}$ and $\mu(t) = 2\sqrt{t}+1$. From equation (2.2), we obtain for $t \in \mathbb{T} = \mathbb{N}^2$

$$\begin{aligned} u(t) &= e_{\frac{\frac{1+\sqrt{t}}{\sqrt{t}}-1}{\mu(t)}}(t,1)1 \\ &= e_{\frac{\frac{1+\sqrt{t}}{\sqrt{t}}-1}{2\sqrt{t}+1}}(t,1) \\ &= e_{\frac{1}{2t+\sqrt{t}}}(t,1). \end{aligned}$$

Consider the original equation $u^\sigma = \frac{1+\sqrt{t}}{\sqrt{t}}u$. By plugging in the appropriate values of t we find

$$u(2^2) = \frac{1+1}{1} = 2$$

$$u(3^2) = \frac{1+2}{2}2 = 3$$

$$u(4^2) = \frac{1+3}{3}3 = 4$$

$$u(5^2) = \frac{1+4}{4}4 = 5.$$

In general,

$$u(t) = \sqrt{t}.$$

Hence,

$$e_{\frac{1}{2t+\sqrt{t}}}(t,1) = \sqrt{t}.$$

Example 3.6. Solve the IVP

$$u^\sigma - \frac{2t^2+1}{2t^2-1}u = 0, \quad y(1) = 1, \tag{3.6}$$

on the time scale $\mathbb{T} = \mathbb{N}^{\frac{1}{2}} = \{t = n^{\frac{1}{2}} | \ n \in \mathbb{N}\}$. In this case $p(t) = \frac{2t^2+1}{2t^2-1}$. From equation (2.2), we obtain for $t \in \mathbb{T} = \mathbb{N}^{\frac{1}{2}}$

$$\begin{aligned} u(t) &= e_{\frac{\frac{2t^2+1}{2t^2-1}-1}{\mu(t)}}(t,1)(1) \\ &= e_{\frac{2t^2+1-2t^2+1}{(2t^2-1)\mu(t)}}(t,1) \\ &= e_{\frac{2}{(2t^2-1)\mu(t)}}(t,1). \end{aligned}$$

But consider the original equation $u^\sigma(t) = \frac{2t^2+1}{2t^2-1}u(t)$. By plugging in the appropriate values of t we find

$$u(2^{\frac{1}{2}}) = \frac{2(1)+1}{2(1)-1}(1) = 3$$

$$u(3^{\frac{1}{2}}) = \frac{2(2)+1}{2(2)-1}(3) = \frac{5}{3}(3) = 5$$

$$u(4^{\frac{1}{2}}) = \frac{2(3)+1}{2(3)-1}(5) = \frac{7}{5}(5) = 7$$

$$u(5^{\frac{1}{2}}) = \frac{2(4)+1}{2(4)-1}(7) = \frac{9}{7}(7) = 9.$$

Hence,

$$e_{\frac{2}{(2t^2-1)\mu}} = 2t^2 - 1.$$

By varying the time and interest rate of an IRA we are able to apply our variation of constants formula (Theorem 2.1) to find out how much money will be in the IRA after time $t \in \mathbb{T}$.

Example 3.7. Suppose we open an IRA account, initially at time $t = 1$, invest \$2,000 and add an additional \$2,000 every time interest is compounded. Let $\mathbb{T} = q^{\mathbb{N}_0}$ where $t_0 = q^0 = 1$. The interest is compounded every $t = q^n$ years and at a rate of $t\%$. How much will we have in the IRA at time t? How much would we have at time $t = (1.5)^5$?

Since $y(1) = 0$, let $y(t)$ be the amount of money after time $t = q^n$. So

$$y^\sigma(t) = y(t) + 2000 + .01t(y(t) + 2000)$$

$$y^\sigma(t) = (1 + .01t)y(t) + (1 + .01t)2000$$

$$y^\sigma(t) - (1 + .01t)y(t) = (1 + .01t)2000. \tag{3.7}$$

Hence we get an equation of the form $y^\sigma - p(t)y = r(t)$ where $p(t) = 1 + .01t$ and $r(t) = (1 + .01t)2000$, and we can solve for $y(t)$ by the variation of constants formula given in Theorem 2.1 to obtain

$$y(t) = \int_1^t e_{\frac{(1+.01t)-1}{\mu(t)}}(t, \sigma(\tau)) \frac{(1 + .01\tau)2000}{\mu(\tau)} \Delta\tau.$$

Since $\mathbb{T} = q^{\mathbb{N}_0}$, $\mu(t) = (q-1)t$. Thus

$$\begin{aligned} y(t) &= \int_1^t e_{\frac{.01t}{(q-1)t}}(t, \sigma(\tau)) \frac{(1 + .01\tau)2000}{(q-1)\tau} \Delta\tau \\ &= \frac{2000}{q-1} \int_1^t \frac{1 + .01\tau}{\tau} e_{\frac{.01}{q-1}}(t, \sigma(\tau)) \Delta\tau. \end{aligned}$$

But $\frac{.01}{q-1}$ is a constant and so $e_{\frac{.01}{q-1}}(t, 1) = \prod_{s=1}^{\rho(t)} [1 + (q-1)(\frac{.01}{q-1})s] = \prod_{s=1}^{\rho(t)} [1 + .01s]$ (see page 74 of [1]). Hence

$$y(t) = \frac{2000}{q-1} \int_1^t \frac{1 + .01\tau}{\tau} \prod_{s \in [\sigma(\tau), t)} [1 + .01s] \Delta\tau.$$

Therefore

$$\begin{aligned} y(t) &= \frac{2000}{q-1} \int_1^t \frac{1 + .01\tau}{\tau} \prod_{s \in [\sigma(\tau), t)} [1 + .01s] \Delta\tau \\ &= \frac{2000}{q-1} \sum_{i \in [1,t)} \left(\mu(i) \frac{1 + .01i}{i} \prod_{s \in [\sigma(i), t)} [1 + .01s] \right) \\ &= \frac{2000}{q-1} \sum_{i \in [1,t)} \left((q-1)i \frac{1 + .01i}{i} \prod_{s \in [\sigma(i), t)} [1 + .01s] \right) \\ &= 2000 \sum_{i \in [1,t)} \left((1 + .01i) \prod_{s \in [\sigma(i), t)} [1 + .01s] \right). \end{aligned}$$

So the solution of the IVP for the IRA problem is

$$y(t) = 2000 \sum_{i=1}^{\rho(t)} \left((1 + .01i) \prod_{s=\sigma(i)}^{\rho(t)} [1 + .01s] \right).$$

If $q = 1.5$ and $n = 5$ we calculate how much money is in our IRA after

$t = (1.5)^5 \approx 7.59$, so approximately 6.59 years after our initial investment, we get

$$y((1.5)^5) = 2000 \sum_{i \in [1,(1.5)^5)} (1 + .01i) \prod_{i \in [\sigma(i),(1.5)^5)} [1 + .01s]$$

which is approximately $11,025.71.

Example 3.8. We now consider an example for the Series Solution Theorem 2.3. On the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{0\} \cup \{q^n | \, n \in \mathbb{Z}\}$, this series solution can be useful where $p(t)$ and $r(t)$ in (2.1) are both monomials. On this time scale, we note that

$$r^{\sigma^k}(t) = r(q^k t)$$

and thus advancing a monomial term k times will multiply the term in the original polynomial $r(t)$ by q raised to k times the order of the term, i.e.

$$(\alpha x^\beta)^{\sigma^k} = \alpha(q^k x)^\beta = \alpha x^\beta q^{k\beta}.$$

We consider the case $p(t) = \alpha t^\beta$ and $r(t) = At^c$, where $\alpha, \beta, A, c \in \mathbb{R}$. Thus we find the series solution to $y^\sigma - \alpha t^\beta y = At^c$ for $t \neq 0$ as follows:

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \frac{-r^{\sigma^k}(t)}{p(t)p^\sigma(t)p^{\sigma^2}(t)p^{\sigma^3}(t)\dots p^{\sigma^k}(t)} \\ &= \sum_{k=0}^{\infty} \frac{-Aq^{kc}t^c}{(\alpha t^\beta)(\alpha q^\beta t^\beta)(\alpha q^{2\beta} t^\beta)\dots(\alpha q^{k\beta} t^\beta)} \\ &= \sum_{k=0}^{\infty} \frac{-Aq^{kc}t^c}{\alpha^k q^{(1+2+\ldots+k)\beta} t^{k\beta}} \\ &= \sum_{k=0}^{\infty} \frac{-Aq^{kc}t^c}{\alpha^k q^{\frac{(k+1)k}{2}\beta} t^{k\beta}} \\ &= -At^c \sum_{k=0}^{\infty} \left(\frac{q^c}{\alpha t^\beta q^{\frac{(k+1)}{2}\beta}} \right)^k. \end{aligned}$$

Now, we note that this series converges by the ratio test for all $t \neq 0$ as

$$\lim_{k\to\infty} \left| \frac{\left(\frac{q^c}{q^{\frac{k+2}{2}\beta} \alpha t^\beta} \right)^{k+1}}{\left(\frac{q^c}{q^{\frac{k+1}{2}\beta} \alpha t^\beta} \right)^{k}} \right| = \lim_{k\to\infty} \left| \frac{q^c}{\alpha t^\beta} \frac{q^{\frac{k^2+k}{2}\beta}}{q^{\frac{k^2+3k+2}{2}\beta}} \right| = \lim_{k\to\infty} \left| \frac{q^c}{\alpha t^\beta} q^{-(k+1)\beta} \right| = 0.$$

Thus for $t \neq 0$ we have as a solution

$$y(t) = -r(t) \sum_{k=0}^{\infty} \frac{q^{kc}}{q^{\frac{(k+1)k}{2}\beta}(\alpha t^{\beta})^k}$$

for any monomial $r(t)$ on the time scale $\overline{q^{\mathbb{Z}}}$, and for $t = 0$, we note that 0 is a dense point in this time scale and thus $y^{\sigma}(0) = y(0)$, and $y^{\sigma}(t) - \alpha t^{\beta} y(t) = r(t)$ becomes simply $y(0) = r(0)$.

Example 3.9. For the time scale $\mathbb{N}^b = \{n^b | \, n \in N\}$ where $b \in \mathbb{R}, b > 0$, we find that for $p(t)$ a monomial, the corresponding solution to (2.1) is

$$e_{\frac{\alpha t^{\beta}-1}{\mu}}(t, t_0) = \alpha^{n_t} \left(\frac{\Gamma(t^{\frac{1}{b}})}{\Gamma(t_0^{\frac{1}{b}})} \right)^{b\beta}$$

which is a solution to $u^{\sigma}(t) = \alpha t^{\beta} u(t)$, $u(t_0) = 1$ as the reader can easily verify.

Given this solution to the homogeneous problem, we can now solve corresponding nonhomogeneous equations.

Example 3.10. Solve the nonhomogeneous problem

$$y^{\sigma} - \alpha t^{\beta} y = (\alpha t^{\beta}) \alpha^{n_t} \left(\frac{\Gamma(t^{\frac{1}{b}})}{\Gamma(t_0^{\frac{1}{b}})} \right)^{b\beta}.$$

Since from the previous problem we have found the exponential

$$e_{\frac{\alpha t^{\beta}-1}{\mu}}(t, t_0) = \alpha^{n_t} \left(\frac{\Gamma(t^{\frac{1}{b}})}{\Gamma(t_0^{\frac{1}{b}})} \right)^{b\beta}$$

we may use our variation of constants formula (2.4) to find

$$\begin{aligned} y(t) &= \alpha^{n_t} \left(\frac{\Gamma(t^{\frac{1}{b}})}{\Gamma(t_0^{\frac{1}{b}})} \right)^{b\beta} \left(y_0 + \int_{t_0}^{t} \frac{\alpha s^{\beta} e_{\frac{\alpha t^{\beta}-1}{\mu}}(s, t_0)}{\mu(s) \alpha s^{\beta} e_{\frac{\alpha t^{\beta}-1}{\mu}}(s, t_0)} \right) \\ &= e_{\frac{\alpha t^{\beta}-1}{\mu}}(t, t_0) \left(y_0 + \int_{t_0}^{t} \frac{\Delta s}{\mu(s)} \Delta s \right) \\ &= e_{\frac{\alpha t^{\beta}-1}{\mu}}(t, t_0)(y_0 + n_t). \end{aligned}$$

This motivates the following corollary to the variation of constants theorem:

Corollary 3.1. *The solution to the IVP*

$$y^{\sigma} - p(t)y = f(t) e_q(t, t_0), \quad y(t_0) = y_0$$

where $q \in \mathcal{R}$, $p(t) \neq 0$, $\forall t \in \mathbb{T}^\kappa$ is given by

$$y(t) = e_{\frac{p-1}{\mu}}(t, t_0)\left(y_0 + \int_{t_0}^{t} \frac{f(s)e_{q \ominus \frac{p-1}{\mu}}(s, t_0)}{p(s)\mu(s)}\Delta s\right).$$

Proof. By Theorem 2.1, we have the solution of the given IVP is

$$y(t) = e_{\frac{p-1}{\mu}}(t, t_0)\left(y_0 + \int_{t_0}^{t} \frac{f(s)e_q(s, t_0)}{\mu(s)p(s)e_{\frac{p-1}{\mu}}(s, t_0)}\Delta s\right),$$

and since

$$\frac{e_q(t, t_0)}{e_{\frac{p-1}{\mu}}(t, t_0)} = e_q(t, t_0)e_{\ominus \frac{p-1}{\mu}}(t, t_0) = e_{q \oplus (\ominus \frac{p-1}{\mu})}(t, t_0) = e_{q \ominus \frac{p-1}{\mu}}(t, t_0)$$

we have our desired result. □

4. Enumerations

By an enumeration on an isolated time scale, consider the mapping $n_t(\cdot, t_0) : \mathbb{T} \to \mathbb{Z}$ given by the counting function, and name each point of the time scale with an index of the value of the counting function at that point. Thus, our anchor point t_0 stays the same, $\sigma(t)$ becomes t_1, $\sigma^2(t)$ becomes t_2, and so on, again with the convention that negative exponents indicate use of the backward jump operator ρ rather than the forward jump operator σ.

For the problem $y^\sigma = (n_t + c)y$, $c \in \mathbb{R}$, enumerate the time scale and consider the function values as the elements in a sequence, where $t_k = \sigma^k(t)$ and $y_k = y(t_k)$. The solution to this recurrence relation is then

$$y_j = \begin{cases} \frac{\Gamma(j+c)}{\Gamma(c)} y_0, & j \geq 0 \\ \frac{(-1)^j \Gamma(1-c)}{\Gamma(1-c-j)} y_0, & j < 0 \end{cases}.$$

For $c = 0$ on $\mathbb{Z}$, the solution breaks at $t = 0$, but considering the solution on the point $t \leq 0$ we have an alternating sequence of the reciprocals of factorials, and for $t \geq 0$ we have the factorials.

We now consider a generalization of our main problem, namely

$$y^{\sigma^k} - p(t)y = r(t).$$

Viewing this as a recurrence relation, we notice that only every k^{th} point is connected from any given initial value; for $k = 2$, this formula would only relate points of even index to points of even index, and points of odd index to points of odd index: the solutions on these two sets of points will be independent of each other.

Theorem 4.1. ***Recurrence Solution on Partitioned Time Scale*** *Consider the recurrence relation*

$$p_1(t)y^{\sigma^{a_1}} + p_2(t)y^{\sigma^{a_2}} + \ldots + p_n(t)y^{\sigma^{a_n}} + q(t)y = r(t), q(t) \neq 0 \tag{4.1}$$

and let $g = gcd\{a_i\}, k = max\{a_i\}$. *Partition the time scale* $\mathbb{T}$ *into the time scales* $\mathbb{T}_n := \{t|\, n_t(t, t_0) \equiv n(mod\, g)\}$ *for* $0 \leq n < g$. *Then solving this recurrence relation is equivalent to solving an IVP of order* $\frac{k}{g}$ *on each of the time scales* $\mathbb{T}_n$ *where the function* μ *is replaced by* $\tilde{\mu} := \mu + \mu^{\sigma} + \mu^{\sigma^2} + \ldots + \mu^{\sigma^{g-1}}$.

Proof. First, we note that the function values y_p, y_q are related by this recurrence relation iff

$$\sigma^{r_1}(t_p) = \sigma^{r_2}(t_q)$$

for some r_1, r_2 which are linear combinations of the a_i. To see this, we note that by looking at (4.1) with $t = t_p$ that y_p must obey a relationship with all points y_{p+a_i}, and then repeatedly applying (4.1) yields that y_p is related to values $y_{p+c_1a_1+c_2a_2+\ldots+c_na_n}$, where the c_i's are integer constants, including negative integers from looking back along the time scale. Now, if there is some point t_q such that

$$y_{q+d_1a_1+d_2a_2+\ldots+d_na_n} = y_{p+c_1a_1+c_2a_2+\ldots+c_na_n}$$

then these values are related through (4.1). Since the smallest positive value of a linear combination of the a_i's is g then related function values can be no less than g jumps apart, and since there exists some linear combination of the a_i with sum g then points g jumps apart are related by our formula. Thus, we must have g sets of function values for y which are not related to one another by the equation, and these independent sets of solution points are the values on the time scales $\mathbb{T}_n$. Now, since one use of the jump operator on $\mathbb{T}_n$ is equivalent to g uses of the jump operator on $\mathbb{T}$, the order of the recurrence relation (4.1) is reduced by a factor of g on all time scales $\mathbb{T}_n$. The converse is obvious. □

Thus we find that the solution to $y^{\sigma^k} - p(t)y = r(t), p \neq 0$ is

$$y([\mathbb{T}]) = \bigcup_{n \in \{0,1,2,\ldots,g-1\}} y([\mathbb{T}_n])$$

where

$$y|_{\mathbb{T}_n} = e_{\frac{p-1}{\tilde{\mu}}}(t, t_n) \left(y_n + \int_{t_n}^{t} \frac{r(s)}{\tilde{\mu}(s)p(s)e_{\frac{p-1}{\tilde{\mu}}}(s, t_n)} \Delta s \right).$$

Example 4.1. Solve the sixth order recurrence relation

$$y^{\sigma^6} - y^{\sigma^3} - y = 0 \tag{4.2}$$

on the time scale $q^{\mathbb{N}_0}$ with $t_0 = 1$ and initial conditions

$$y(1) = 1,\ y(q) = -5,\ y(q^2) = 8 \times 10^{10^2},$$
$$y(q^3) = 1,\ y(q^4) = -9,\ y(q^5) = 13 \times 10^{10^2}.$$

Noting that $\gcd\{3,6\} = 3$, we first partition $q^{\mathbb{N}_0}$ into three time scales using the counting function $n_t(t, t_0) = \log_q(t/t_0) = \log_q(t)$, forming the cells $\mathbb{T}_0 = q^{3\mathbb{N}_0}$, $\mathbb{T}_1 = q^{3\mathbb{N}_0+1}$, and $\mathbb{T}_2 = q^{3\mathbb{N}_0+2}$. On each of these time scales we consider the recurrence relation $y^{\sigma^2} - y^{\sigma} - y = 0$, which we recognize as the relation generating the Fibonacci sequence. Thus, on $\mathbb{T}_0$ with initial values 1 and 1, we have the classic Fibonacci sequence, on $\mathbb{T}_1$ the initial conditions -5 and -9 will give a Fibonacci type sequence with all terms negative, and on $\mathbb{T}_2$, our initial conditions $y(q^2) = 8 \times 10^{10^2}$, $y(q^5) = 13 \times 10^{10^2}$ will give Fibonacci sequence values multiplied by one googol.

5. Recurrence Relations and the Counting Function

Note that for $f(t) \neq 0$ for all $t \in \mathbb{T}$ that for

$$u^{\sigma} = \frac{f^{\sigma}}{f} u \tag{5.1}$$

a general solution to (5.1) is $u(t) = cf(t)$. Now, in particular, if in (5.1) we consider $f(n_t)$ rather than $f(t)$ then we can view the problem as a simple relation between functions of consecutive integers or terms in a sequence, as $f^{\sigma}(n_t) = f(n_t + 1)$. Now, we may therefore approach problems of the form $y^{\sigma} = p(t)y + r(t)$ by converting $p(t)$ into a rational function of the form $\frac{f(n_t+1)}{f(n_t)}$. We note that any isolated time scale can be enumerated by use of the counting function

$$n_t(t, t_0) = \int_{t_0}^{t} \frac{\Delta s}{\mu(s)}.$$

Using this approach, we find that many of our examples can be solved an easier way:

$$u^{\sigma} = qu = \frac{q^{n_t+1}}{q^{n_t}} u \Longrightarrow u = cq^{n_t}$$

with $q^{n_t} = \frac{t}{t_0}$ on $\overline{q^{\mathbb{Z}}}, t, t_0 \neq 0$, as Example 3.1,

$$u^{\sigma} = \frac{\mu(t)}{\mu^{\sigma}(t)} u \Longrightarrow u = \frac{c}{\mu}$$

with $\frac{1}{\mu(t)} = n + 1$ on $\{H_n\}_0^{\infty}$, as in Example 3.2,

$$u^{\sigma} = \frac{2 + n_t}{1 + n_t} u \Longrightarrow u = c(n_t + 1)$$

and in particular, on $\mathbb{N}^2$ where $n_t(t,1) = \sqrt{t} - 1$

$$u^\sigma = \frac{1+\sqrt{t}}{\sqrt{t}} u \Longrightarrow u = c\sqrt{t}$$

as in Example 3.5, and on $\mathbb{N}^{\frac{1}{2}}$, with $n_t(t,1) = t^2 - 1$ we see that

$$u^\sigma = \frac{2t^2+1}{2t^2-1} u = \frac{(2n_t(t,1)+1)^\sigma}{2n_t(t,1)+1} u \Longrightarrow u = c(2n_t(t,1)+1)$$

as in Example 3.6. This seems to suggest that it may be of value to consider recurrence relations on time scales as sequences. The closure of the image of the interior of any isolated interval of a time scale under a map which is increasing and injective will again be a time scale, as the image of the open interval between points will be an open interval between points in the image. Thus any portion of a sequence can be transformed to be on any isolated interval of a time scale, so long as the cardinalities match. The counting function always provides a map back to the integers, which seems to suggest that understanding the transformations between time scales depends on studying the lengths of the images of intervals.

References

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