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Oscillation Theorems for a Self-Adjoint Dynamic Equation on Time Scales

Aaron Gray, David Lovit and Steven Spriggs
University of Nebraska-Lincoln
Department of Mathematics
Lincoln, NE 68588-0323, USA
gray0150@umn.edu
dlovit1@swarthmore.edu
spriggss@shawnee.edu

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Abstract

We obtain several oscillation theorems for self-adjoint second-order mixed-derivative linear dynamic equations on time scales. Wintner, Erbe–Peterson, and Leighton–Wintner type oscillation theorems and a Hille–Wintner comparison type theorem are obtained for this mixed equation. Several examples are given.

1. Introduction

We will be concerned with proving several oscillation theorems for the formally self-adjoint second-order linear dynamic equation

$$\big(p(t)x^{\Delta}\big)^{\nabla} + q(t)x = 0. \tag{1.1}$$

Some analagous results for the equation

$$\big(p(t)x^{\Delta}\big)^{\Delta} + q(t)x^{\sigma} = 0 \tag{1.2}$$

have already been proven [2], [4]. Since corresponding to equation (1.1) one can define a self-adjoint operator in the functional analysis sence, many researchers prefer equation (1.1) to equation (1.2). Also equation (1.1) is preferred because certain Green's functions corresponding to (1.1) turn out to be symmetric [1].

For completeness, we recall the following concepts related to the notion of time scales. A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The **forward jump operator** and the **backward jump operator** are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$ and $\inf \emptyset = \sup \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be **left–dense** if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is **right–dense** if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, is **left–scattered** if $\rho(t) < t$ and **right–scattered** if $\sigma(t) > t$. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa := \mathbb{T} \backslash \{m\}$, with $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. Similarly, if $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa := \mathbb{T} \backslash \{M\}$, with $\mathbb{T}^\kappa = \mathbb{T}$ otherwise.

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be **right–dense continuous** (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$. Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is said to be **left–dense continuous** (ld–continuous) provided f is continuous at left–dense points and at right–dense points in $\mathbb{T}$, right hand limits exist and are finite. The set of all such ld–continuous functions on $\mathbb{T}$ is denoted by $C_{ld}(\mathbb{T})$. The **(forward) graininess function** μ and the **backwards graininess function** ν for a time scale $\mathbb{T}$ are defined by

$$\mu(t) = \sigma(t) - t, \quad \nu(t) := t - \rho(t),$$

and for any function $f : \mathbb{T}^\kappa \to \mathbb{R}$ and any function $g : \mathbb{T}_\kappa \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$ and the notation $g^\rho(t)$ denotes $g(\rho(t))$.

The following definitions and theorems, found in [3], have analogous results corresponding to the delta derivative, found in [2].

Definition 1.1. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^\nabla(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[x(\rho(t)) - x(s)] - x^\nabla(t)[\rho(t) - s]| \leq \epsilon|\rho(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^\nabla(t)$ is the **nabla derivative** of x at t and that x is **nabla differentiable** at t.

The following theorem is important when studying nabla derivatives (see [1] and [2, Theorem 8.41]).

Theorem 1.2. *Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}_\kappa$.*
(i) If g is nabla differentiable at t, then g is continuous at t.
(ii) If g is continuous at t and t is left-scattered, then g is nabla differentiable at t with

$$g^\nabla(t) = \frac{g(t) - g(\rho(t))}{\nu(t)}.$$

(iii) If g is nabla differentiable and t is left-dense, then

$$g^\nabla(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is nabla differentiable at t, then $g(\rho(t)) = g(t) - \nu(t)g^{\nabla}(t)$.
(v) If f and g are nabla differentiable at t, then

$$(fg)^{\nabla}(t) = f^{\rho}(t)\, g^{\nabla}(t) + f^{\nabla}(t)\, g(t) = f(t)\, g^{\nabla}(t) + f^{\nabla}(t)\, g^{\rho}(t).$$

(vi) If f and g are nabla differentiable at t and $g(t)\, g^{\rho}(t) \neq 0$, then

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)\, g(t) - f(t)\, g^{\nabla}(t)}{g(t)\, g^{\rho}(t)}.$$

Definition 1.3. If $G^{\nabla}(t) = g(t)$, then the **Cauchy (nabla) integral** is defined by

$$\int_a^t g(s)\nabla s := G(t) - G(a).$$

The following [3, Theorem 4.4] is a generalization of L'Hôpital's rule for ∇ derivatives.

Theorem 1.4 (L'Hôpital's Rule)**.** *Assume f and g are ∇ differentiable on $\mathbb{T}$ and let $t_0 \in \mathbb{T}$, and assume t_0 is right-dense. Furthermore, assume*

$$\lim_{t\to t_0^+} f(t) = \lim_{t\to t_0^+} g(t) = 0$$

and suppose there exists $\varepsilon > 0$ with

$$g(t)\, g^{\nabla}(t) > 0 \quad \text{for all} \quad t \in R_{\varepsilon}(t_0).$$

Then

$$\liminf_{t\to t_0^+} \frac{f^{\nabla}(t)}{g^{\nabla}(t)} \le \liminf_{t\to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t\to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t\to t_0^+} \frac{f^{\nabla}(t)}{g^{\nabla}(t)}.$$

The following important result appears in Atici and Gusinov [1] and has been generalized by Messer in [3, Theorem 4.8].

Theorem 1.5. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ differentiable on $\mathbb{T}^{\kappa}$ and if f^{Δ} is continuous on $\mathbb{T}^{\kappa}$, then f is ∇ differentiable on $\mathbb{T}_{\kappa}$ and $f^{\nabla} = f^{\Delta\rho}$ on $\mathbb{T}_{\kappa}$.*

2. Preliminary Results

We consider the formally self-adjoint equation (1.1) with mixed derivatives, where p, q are continuous and $p(t) > 0$ for all $t \in \mathbb{T}$. We define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous and $\left(px^{\Delta}\right)^{\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is continuous. A function $x \in \mathbb{D}$ is then said to be a solution of (1.1) on $\mathbb{T}$ provided $(p(t)x^{\Delta}(t))^{\nabla} + q(t)x(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

The following three results are proven in [1].

Theorem 2.1 (Existence and Uniqueness)**.** *If f is a continuous function of t, $t_0 \in \mathbb{T}^{\kappa}_{\kappa}$, and x_0, x_0^{Δ} are given constants, then the initial value problem*

$$Lx := (p(t)x^{\Delta}(t))^{\nabla} + q(t)x(t) = f(t), \quad x(t_0) = x_0, \quad x^{\Delta}(t_0) = x_0^{\Delta}$$

has a unique solution that exists on the set $\mathbb{T}$.

Definition 2.2. The **Wronskian** $W(x,y)$ of two differentiable functions x and y is defined by

$$W(x,y)(t) := \begin{vmatrix} x(t) & y(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{vmatrix}, \quad t \in \mathbb{T}^{\kappa}.$$

If x and y are linearly dependent, then it follows that $W(x,y) \equiv 0$. The next few results show that $W(x,y)$ is either always 0 or never 0 for any pair of solutions x and y of (1.1), so it can be used to determine whether or not two solutions are linearly independent.

Definition 2.3. The **Lagrange bracket** $\{x;y\}$ of two functions x and y is

$$\{x;y\} = pW(x,y)$$

Lemma 2.4 (Lagrange identity)**.** *If $x, y \in \mathbb{D}$, then*

$$\{x;y\}^{\nabla}(t) = x(t)Ly(t) - y(t)Lx(t), \tag{2.1}$$

for $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Lemma 2.5 (Abel's formula)**.** *If x and y are solutions of* (1.1)*, then*

$$W(x,y)(t) = \frac{C}{p(t)}$$

for all $t \in \mathbb{T}^{\kappa}$ where C is a constant.

For any two solutions x and y of (1.1), $W(x,y) \equiv 0$ iff x and y are linearly dependent on $\mathbb{T}$ and $W(x,y) \neq 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$ iff x and y are linearly independent.

Definition 2.6. Assume $x : \mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(t) \neq 0$, then the **Riccati substitution** is

$$z(t) = \frac{p(t)x^{\Delta}(t)}{x(t)} \tag{2.2}$$

for $t \in \mathbb{T}^{\kappa}$.

A continuous function on a time scale may change sign without ever assuming the value of zero, which leads to the following.

Definition 2.7. A function $x : \mathbb{T} \to \mathbb{R}$ has a **generalized zero** at t provided $x(t) = 0$ and if $\rho(t) < t$ we say x has a generalized zero in $(\rho(t), t)$ if

$$p(\rho(t))\, x(\rho(t))\, x(t) < 0.$$

Throughout this paper we assume

$$\omega := \sup \mathbb{T}$$

and if $\omega < \infty$, $\mathbb{T}$ is a time scale such that $\rho(\omega) = \omega$. In this last case we do not assume that the coefficient functions in $Lx = 0$ are defined at ω (so ω is a singular point). Let $b \in \mathbb{T}$ with $b < \omega$, then we say that (1.1) is **oscillatory** on $[b, \omega)$ if every nontrivial real-valued solution has infinitely many generalized zeros in $[b, \omega)$. Otherwise (1.1) is **nonoscillatory** on $[b, \omega)$.

The next result [3, Theorem 4.58] is central to the proof of several oscillation theorems.

Theorem 2.8. *If x is a solution of* (1.1) *with no generalized zeros in $\mathbb{T}$, then z as defined by* (2.2) *for $t \in \mathbb{T}^\kappa$ is a solution to the* **Riccati equation**

$$Rz := z^\nabla + q + \frac{(z^\rho)^2}{p^\rho + \nu z^\rho} = 0 \tag{2.3}$$

on $\mathbb{T}^\kappa$ and

$$p^\rho(t) + \nu(t) z^\rho(t) > 0 \tag{2.4}$$

for $t \in \mathbb{T}^\kappa_\kappa$.

The following two results [3, Theorems 4.49 and 4.52] pertain to the factorization of (1.1) under certain conditions.

Theorem 2.9 (Polya factorization)**.** *If* (1.1) *has a positive solution u on $\mathbb{T}$, then for $x \in \mathbb{D}$,*

$$Lx(t) = \psi_1(t)\left(\psi_2(t)\left(\psi_1(t)x(t)\right)^\Delta\right)^\nabla, \quad t \in \mathbb{T}^\kappa_\kappa, \tag{2.5}$$

where

$$\psi_1(t) := \frac{1}{u(t)} > 0, \quad t \in \mathbb{T} \quad \text{and} \quad \psi_2(t) := p(t)u(t)u^\sigma(t) > 0, \quad t \in \mathbb{T}^\kappa_\kappa.$$

Theorem 2.10 (Trench factorization)**.** *If* (1.1) *has a Polya factorization on $[a, \omega)$, then* (1.1) *has a Polya factorization with*

$$\int_a^\omega \frac{1}{\psi_2(s)} \Delta s = +\infty,$$

which is called a Trench factorization of (1.1) *on $[a, \omega)$.*

The following result [3, Theorem 4.45] elaborates on the properties of a specific solution provided by the factorizations.

Theorem 2.11 (Recessive and Dominant Solutions)**.** *If* (1.1) *has a Trench factorization on $[a, \omega)$, then the solution $u = \frac{1}{\psi_1}$ satisfies*

$$\int_a^\omega \frac{1}{p(t)\, u(t)\, u^\sigma(t)} \Delta t = +\infty$$

and for any linearly independent solution v,

$$\lim_{t \to \omega} \frac{u(t)}{v(t)} = 0$$

Also, $\exists\, b \in [a, \omega)$ such that

$$\int_b^\omega \frac{1}{p(t)\, v(t)\, v^\sigma(t)} \Delta t < +\infty$$

and for all $t \in [b, \omega)$,

$$\frac{p(t)\, v^{\Delta}(t)}{v(t)} > \frac{p(t)\, u^{\Delta}(t)}{u(t)}.$$

We call u *a* **recessive solution** *of* (1.1) *at* ω*, and the above properties of* u *make it unique up to multiplication by a nonzero constant. Any linearly independent solution* v *is called a* **dominant solution** *at* ω*.*

The following lemma (for an analagous result see [5, Lemma 13] and [4, Lemma 1.4]) is used in the proof of the Hille–Wintner theorem.

Lemma 2.12. *Assume*

$$\liminf_{t\to\omega} \int_T^t q(s)\, \nabla s \geq 0 \text{ and not } \equiv 0 \tag{2.6}$$

for all large T*, and*

$$\int_a^\omega \frac{1}{p(s)} \Delta s = \infty. \tag{2.7}$$

If x *is a solution of* (1.1) *such that* $x(t) > 0$ *for* $t \in [T, \omega)$*, then there exists* $S \in [T, \omega)$ *such that* $x^{\Delta}(t) > 0$ *for* $t \in [S, \omega)$*.*

Equivalent theorems to the following two appear in [3, Theorems 4.66 and 4.68].

Theorem 2.13. *If the Riccati dynamic inequality* $Rz \leq 0$ *has a solution on* $[a, \omega)$*, then* $Lx = 0$ *is nonoscillatory on* $[a, \omega)$*.*

Theorem 2.14 (Sturm Comparison Theorem)**.** *Suppose we have two equations of the same form as* (1.1),

$$L_1 x = \left(p_1 x^{\Delta}\right)^{\nabla} + q_1 x = 0$$

$$L_2 x = \left(p_2 x^{\Delta}\right)^{\nabla} + q_2 x = 0$$

such that $q_2 \leq q_1$ *and* $0 < p_1 \leq p_2$ *on* $[a, \omega)$*. Then if* $L_1 x = 0$ *is nonoscillatory on* $[a, \omega)$*, then* $L_2 x = 0$ *is nonoscillatory on* $[a, \omega)$*.*

3. Main Results

Theorem 3.1 (Wintner's Theorem)**.** *Assume* $\sup \mathbb{T} = \infty$, $a \in \mathbb{T}$*, and there exist constants* K *and* M *such that* $\nu(t) \geq K > 0$ *and* $M \geq p(t) > 0$ *on* $[a, \infty)$*, and furthermore*

$$\int_a^\infty q(t) \nabla t = +\infty.$$

Then (1.1) *is oscillatory on* $[a, \infty)$*.*

Proof. Assume (1.1) is nonoscillatory on $[a,\infty)$, then there is a solution x that does not have infinitely many generalized zeros on $[a,\infty)$. Then there is a $t_0 \in (a,\infty)$ such that x has no generalized zeros on $[\rho(t_0),\infty)$. We then perform the Riccati substitution (2.2) to obtain a solution z to (2.3) satisfying (2.4) on $[t_0,\infty)$. Then for $t \in [t_0,\infty)$

$$z^{\nabla}(t) = -q(t) - \frac{(z^{\rho}(t))^2}{p^{\rho}(t) + \nu(t)z^{\rho}(t)}.$$

It follows that

$$\begin{aligned} \int_{t_0}^{t} z^{\nabla}(s)\nabla s &= -\int_{t_0}^{t} q(s)\nabla s - \int_{t_0}^{t} \frac{(z^{\rho}(s))^2}{p^{\rho}(s)+\nu(s)z^{\rho}(s)}\nabla s \\ z(t) - z(t_0) &\leq -\int_{t_0}^{t} q(s)\nabla s \\ z(t) &\leq z(t_0) - \int_{t_0}^{t} q(s)\nabla s, \end{aligned}$$

where we have used (2.4) to obtain the inequality. We then have

$$\lim_{t\to\infty} z(t) = -\infty.$$

However

$$p^{\rho}(t) + \nu(t)z^{\rho}(t) > 0 \Rightarrow z^{\rho}(t) > -\frac{p^{\rho}(t)}{\nu(t)} \geq -\frac{M}{K},$$

which is a contradiction. □

Theorem 3.2. *Assume* $\forall\ t_0 \in [a,\omega)$, $\exists\ a_0 \in (t_0,\omega)$, $b_0 \in (a_0,\omega)$ *such that* $\nu(a_0) > 0$, $\nu(b_0) > 0$ *and*

$$\int_{\rho(a_0)}^{\rho(b_0)} q(s)\nabla s \geq \frac{p^{\rho}(a_0)}{\nu(a_0)} + \frac{p^{\rho}(b_0)}{\nu(b_0)}.$$

Then (1.1) *is oscillatory on* $[a,\omega)$.

Proof. Assume (1.1) is nonoscillatory on $[a,\omega)$, then there is a solution x that does not have infinitely many generalized zeros on $[a,\omega)$. Then there is a $t_0 \in (a,\omega)$ such that x has no generalized zeros on $[\rho(t_0),\omega)$. We then use (2.2) to obtain a solution z to (2.3) satisfying (2.4) on $[t_0,\omega)$. Then for any $a_0 \in (t_0,\omega)$, $b_0 \in (a_0,\omega)$ we get the following:

$$\begin{aligned} z^{\nabla}(t) &= -q(t) - \frac{(z^{\rho}(t))^2}{p^{\rho}(t)+\nu(t)z^{\rho}(t)} \\ \int_{\rho(a_0)}^{\rho(b_0)} z^{\nabla}\nabla s &= -\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s - \int_{\rho(a_0)}^{\rho(b_0)} \frac{(z^{\rho})^2}{p^{\rho}+\nu z^{\rho}}\nabla s \\ z^{\rho}(b_0) - z^{\rho}(a_0) &\leq -\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s - \int_{\rho(a_0)}^{a_0} \frac{(z^{\rho})^2}{p^{\rho}+\nu z^{\rho}}\nabla s \end{aligned}$$

$$\begin{aligned}
z^{\rho}\left(b_0\right) &\leq z^{\rho}\left(a_0\right)-\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s-\frac{\nu\left(a_0\right)\left(z^{\rho}\left(a_0\right)\right)^2}{p^{\rho}\left(a_0\right)+\nu\left(a_0\right) z^{\rho}\left(a_0\right)} \\
z^{\rho}\left(b_0\right) &\leq \frac{p^{\rho}\left(a_0\right) z^{\rho}\left(a_0\right)}{p^{\rho}\left(a_0\right)+\nu\left(a_0\right) z^{\rho}\left(a_0\right)}-\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s \\
z^{\rho}\left(b_0\right) &< \frac{p^{\rho}\left(a_0\right)}{\nu\left(a_0\right)}-\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s \\
\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s &< \frac{p^{\rho}\left(a_0\right)}{\nu\left(a_0\right)}-z^{\rho}\left(b_0\right) \\
\int_{\rho(a_0)}^{\rho(b_0)} q\nabla s &< \frac{p^{\rho}\left(a_0\right)}{\nu\left(a_0\right)}+\frac{p^{\rho}\left(b_0\right)}{\nu\left(b_0\right)},
\end{aligned}$$

which is the desired contradiction. □

Theorem 3.3. *Assume* $\sup \mathbb{T}=\infty$*,* $a\in\mathbb{T}$*,* $p\equiv 1$*, and* $\forall\ t_0\in[a,\infty)$*,* $\exists\ \{t_k\}_{k=1}^{\infty}\subset[t_0,\infty)$*,* t_k *strictly increasing and* $\lim_{k\to\infty} t_k=\infty$*. Additionally, assume that* $\exists\ K_1,\ K_2$ *such that* $0<K_1\leq\nu\left(t_k\right)\leq K_2$ *for all* $k\in\mathbb{N}$ *and*

$$\lim_{k\to\infty}\int_{\rho(t_1)}^{\rho(t_k)} q(s)\nabla s\geq\frac{1}{\nu\left(t_1\right)}.$$

Then (1.1) *is oscillatory on* $[a,\infty)$*.*

Proof. Assume (1.1) is nonoscillatory on $[a,\infty)$, then there is a solution x that does not have infinitely many generalized zeros on $[a,\infty)$. Then there is a $t_0\in(a,\infty)$ such that x has no generalized zeros on $[\rho(t_0),\infty)$. Without loss of generality we can assume $x(t)>0$ on $[t_0,\infty)$. We then perform the Riccati substitution to obtain a solution z to (2.3) satisfying (2.4) on $[t_0,\infty)$. Then for any $k\in\mathbb{N}$

$$\begin{aligned}
z^{\nabla} &= -q-\frac{\left(z^{\rho}\right)^2}{1+\nu z^{\rho}} \\
\int_{\rho(t_1)}^{\rho(t_k)} z^{\nabla}\nabla s &= -\int_{\rho(t_1)}^{\rho(t_k)} q\nabla s-\int_{\rho(t_1)}^{\rho(t_k)}\frac{\left(z^{\rho}\right)^2}{1+\nu z^{\rho}}\nabla s \\
z^{\rho}\left(t_k\right)-z^{\rho}\left(t_1\right) &\leq -\int_{\rho(t_1)}^{\rho(t_k)} q\nabla s-\int_{\rho(t_1)}^{t_1}\frac{\left(z^{\rho}\right)^2}{1+\nu z^{\rho}}\nabla s \\
z^{\rho}\left(t_k\right) &\leq z^{\rho}\left(t_1\right)-\int_{\rho(t_1)}^{\rho(t_k)} q\nabla s-\frac{\nu\left(t_1\right)\left(z^{\rho}\left(t_1\right)\right)^2}{1+\nu\left(t_1\right) z^{\rho}\left(t_1\right)} \\
z^{\rho}\left(t_k\right) &\leq \frac{z^{\rho}\left(t_1\right)}{1+\nu\left(t_1\right) z^{\rho}\left(t_1\right)}-\int_{\rho(t_1)}^{\rho(t_k)} q\nabla s
\end{aligned}$$

We will now assume that $\lim_{k\to\infty} z^{\rho}(t_k) = 0$, and shortly produce a contradiction. The proof of this assumption then follows.

$$\begin{aligned}
\lim_{k\to\infty} z^{\rho}(t_k) &\le \lim_{k\to\infty} \frac{z^{\rho}(t_1)}{1+\nu(t_1) z^{\rho}(t_1)} - \lim_{k\to\infty} \int_{\rho(t_1)}^{\rho(t_k)} q\nabla s \\
0 &\le \frac{z^{\rho}(t_1)}{1+\nu(t_1) z^{\rho}(t_1)} - \frac{1}{\nu(t_1)} \\
0 &\le \nu(t_1) z^{\rho}(t_1) - 1 - \nu(t_1) z^{\rho}(t_1) \\
0 &\le -1.
\end{aligned}$$

We thus have a contradiction, so (1.1) must be oscillatory on $[a,\infty)$.

It remains to prove our previous claim that

$$\lim_{k\to\infty} z^{\rho}(t_k) = 0.$$

To see this let

$$F(t) := \frac{(z^{\rho}(t))^2}{1+\nu(t) z^{\rho}(t)} = -z^{\nabla}(t) - q(t).$$

¿From $\lim_{k\to\infty} \int_{\rho(t_1)}^{\rho(t_k)} q(s)\nabla s \ge \frac{1}{\nu(t_1)}$ we know $\exists\, M$ such that $\forall\, k$, $\int_{\rho(t_1)}^{\rho(t_k)} q(s)\nabla s > M$. Now consider

$$\begin{aligned}
\sum_{j=2}^{k-1} F(t_j)\,\nu(t_j) &= \sum_{j=2}^{k-1} \int_{\rho(t_j)}^{t_j} F(t)\,\nabla t \\
&\le \int_{t_1}^{\rho(t_k)} F(t)\,\nabla t \\
&= \int_{\rho(t_1)}^{\rho(t_k)} F(t)\,\nabla t - \int_{\rho(t_1)}^{t_1} F(t)\,\nabla t \\
&= -\int_{\rho(t_1)}^{\rho(t_k)} z^{\nabla}(t)\,\nabla t - \int_{\rho(t_1)}^{\rho(t_k)} q(t)\,\nabla t - \nu(t_1) F(t_1) \\
&= -z^{\rho}(t_k) + z^{\rho}(t_1) - \int_{\rho(t_1)}^{\rho(t_k)} q(t)\,\nabla t - \frac{\nu(t_1)\,(z^{\rho}(t_1))^2}{1+\nu(t_1) z^{\rho}(t_1)} \\
&= -z^{\rho}(t_k) + \frac{z^{\rho}(t_1)}{1+\nu(t_1) z^{\rho}(t_1)} - \int_{\rho(t_1)}^{\rho(t_k)} q(t)\,\nabla t \\
&\le \frac{1}{\nu(t_k)} + \frac{1}{\nu(t_1)} - M \\
&\le \frac{2}{K_1} - M.
\end{aligned}$$

So the series $\sum_{j=2}^{\infty} F(t_j)\nu(t_j)$ converges, and thus $\lim_{k\to\infty} F(t_k)\,\nu(t_k) = 0$. Additionally, as $\nu(t_k) \ge K_1$, $\lim_{k\to\infty} F(t_k) = 0$. Since $0 < K_1 \le \nu(t_k) \le K_2$

for $k \in \mathbb{N}$, we have that

$$F(t_k) = \frac{(z^\rho(t_k))^2}{1+\nu(t_k)z^\rho(t_k)} \geq \frac{(z^\rho(t_k))^2}{1+Kz^\rho\left(t_{k_j}\right)} > 0,$$

where $K = K_2$ if $z^\rho(t_k) \geq 0$ and $K = K_1$ if $z^\rho(t_k) < 0$. Then

$$\lim_{j\to\infty} \frac{z^\rho\left(t_k\right)^2}{1+Kz^\rho\left(t_k\right)} = 0$$

which implies that

$$\lim_{k\to\infty} z^\rho\left(t_k\right) = 0.$$

□

Theorem 3.4 (Leighton–Wintner Theorem)**.** *If*

$$\int_a^\omega \frac{1}{p(t)}\Delta t = \int_a^\omega q(t)\nabla t = +\infty,$$

then (1.1) *is oscillatory on* $[a,\omega)$*.*

Proof. Assume (1.1) is nonoscillatory on $[a,\omega)$, then by Theorem 2.11 there is a dominant solution x of (1.1) with finitely many generalized zeros on $[a,\omega)$, so that for some $T \in [a,\omega)$, x has no generalized zeros on $[\rho(T),\omega)$. Also

$$\int_T^\omega \frac{1}{p(t)x(t)x^\sigma(t)}\Delta t < +\infty.$$

If we let $z(t) = p(t)\frac{x^\Delta(t)}{x(t)}$, $t \in [\rho(T),\omega)$, then from Theorem 2.8 we have $p^\rho(t) + \nu(t)z^\rho(t) > 0$ and

$$z^\nabla = -q(t) - \frac{(z^\rho(t))^2}{p^\rho(t)+\nu(t)z^\rho(t)} \quad \leq \quad -q(t)$$

on $[T,\omega)$. It follows that

$$z(t) \leq z\left(T\right) - \int_T^t q(s)\nabla s, \quad t \in [T,\omega)$$

which implies that

$$\lim_{t\to\omega} z(t) = -\infty.$$

Then $\exists\ T_1 \in [T,\omega)$ such that $z(t) = p(t)\frac{x^\Delta(t)}{x(t)} < 0$ on $[T_1,\omega)$. If $x(t) > 0$ on $[T_1,\omega)$, then x is a positive decreasing function and

$$\int_{T_1}^\omega \frac{1}{p(t)x(t)x^\sigma(t)}\Delta t \geq \frac{1}{x\left(T_1\right)^2}\int_{T_1}^\omega \frac{1}{p(t)}\Delta t = +\infty.$$

Thus we have a contradiction. If $x(t) < 0$, on $[T_1,\omega)$, then x is a negative increasing function, and the same conclusion holds. Thus (1.1) is oscillatory on $[a,\omega)$. □

Next we show that the Hille–Wintner theorem given in Erbe and Peterson [4] for the dynamic equation $(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0$ holds for the mixed dynamic equation (1.1).

Theorem 3.5 (Hille–Wintner Theorem)**.** *Suppose we have two equations of the same form as* (1.1)*,*

$$L_i x = \left(p_i x^{\Delta}\right)^{\nabla} + q_i x = 0, \quad i = 1, 2$$

such that

$$0 < p_1(t) \le p_2(t), \quad t \in [a, \omega) \tag{3.1}$$

$$\int_a^{\omega} \frac{1}{p_1(t)} \Delta t = +\infty \tag{3.2}$$

$$\int_a^{\omega} q_1(t) \nabla t \quad \textit{and} \quad \int_a^{\omega} q_2(t) \nabla t \quad \textit{exist} \tag{3.3}$$

$$0 \le \int_t^{\omega} q_2(s) \nabla s \le \int_t^{\omega} q_1(s) \nabla s, \quad t \in [a, \omega) \tag{3.4}$$

$$\exists\, M > 0 \quad \textit{such that} \quad p_1^{\rho}(t) \le M\nu(t) \quad \textit{provided} \quad \nu(t) > 0. \tag{3.5}$$

Then if $L_1 x = 0$ is nonoscillatory on $[a, \omega)$, then $L_2 x = 0$ is nonoscillatory on $[a, \omega)$.

Proof. Let $\{t_k\}_{k=1}^{\infty} \subset (a, \omega)$ be a strictly increasing sequence of left-scattered points such that $\lim_{k \to \infty} t_k = \omega$. If no such sequence exists, then $\mathbb{T}$ is a real interval past sufficiently large t, and the classic Hille–Wintner theorem (see [6] and [7]) applies. Also, as $L_1 x = 0$ is nonoscillatory, $\exists\, T \in (a, \omega)$ and a solution x of $L_1 x = 0$ with $x(t) > 0$ on $[\rho(T), \omega)$. We then perform a Riccati substitution to get a z such that

$$R_1 z = z^{\nabla} + q_1 + \frac{(z^{\rho})^2}{p_1^{\rho} + \nu z^{\rho}} = 0$$

and

$$p_1^{\rho} + \nu z^{\rho} > 0 \tag{3.6}$$

on $[T, \omega)$. Then if we let

$$F(t) := \frac{\left(z^{\rho}(t)\right)^2}{p_1^{\rho}(t) + \nu(t) z^{\rho}(t)} \ge 0$$

for $t \in [T, \omega)$, we have for $t \ge T$ by integrating both sides of $R_1 z(t) = 0$

$$z(t) + \int_T^t q_1(s) \nabla s + \int_T^t F(s) \nabla s = z(T). \tag{3.7}$$

We have from Lemma 2.12, (3.2), and (3.4) that $z(t) > 0$, so then as $z(T)$ is finite and $\int_T^{\omega} q_1(s) \nabla s$ exists we must have $\int_T^{\omega} F(s) \nabla s < \infty$. Thus $\lim_{t \to \omega} z(t)$

exists. We will now show that $\lim_{t\to\omega} z(t) = 0$. We have from (3.6) and (3.5) that

$$z(t_k) > -\frac{p_1^\rho(t_k)}{\nu(t_k)} \geq -M$$

so using (3.7) with $t = t_k$ we obtain

$$-M + \int_T^{t_k} q_1(s)\nabla s + \int_T^{t_k} F(s)\nabla s \leq z(T).$$

Then if we let n_0 be the first k for which $t_k \geq T$,

$$\sum_{k=n_0}^{\infty} \nu(t_k) F(s)(t_k) = \sum_{k=n_0}^{\infty} \int_{\rho(t_k)}^{t_k} F(s)\nabla s \leq \int_T^{t_k} F(s)\nabla s < \infty.$$

Thus we have

$$\lim_{k\to\infty} \nu(t_k) F(t_k) = \lim_{k\to\infty} \frac{(z^\rho(t_k))^2}{\frac{p_1^\rho(t_k)}{\nu(t_k)} + z^\rho(t_k)} = 0.$$

So for any $\varepsilon > 0$, $\exists$ an integer K such that $k \geq K$ implies

$$\begin{aligned}
\frac{(z^\rho(t_k))^2}{\frac{p_1^\rho(t_k)}{\nu(t_k)} + z^\rho(t_k)} &< \varepsilon \\
(z^\rho(t_k))^2 &< \frac{p_1^\rho(t_k)}{\nu(t_k)}\varepsilon + z^\rho(t_k)\varepsilon \\
(z^\rho(t_k))^2 - z^\rho(t_k)\varepsilon + \frac{\varepsilon^2}{4} &\leq M\varepsilon + \frac{\varepsilon^2}{4} + \sqrt{M\varepsilon^3} \\
\left(z^\rho(t_k) - \frac{\varepsilon}{2}\right)^2 &\leq \left(\sqrt{M\varepsilon} + \frac{\varepsilon}{2}\right)^2 \\
|z^\rho(t_k)| &\leq \sqrt{M\varepsilon} + \varepsilon.
\end{aligned}$$

Thus $\lim_{k\to\infty} z^\rho(t_k) = 0$. Then by continuity and the existence of $\lim_{t\to\infty} z(t)$ we have that

$$\lim_{t\to\omega} z(t) = 0.$$

Using (3.7) we have

$$\int_T^\omega q_1(s)\nabla s + \int_T^\omega F(s)\nabla s = z(T).$$

Now we define

$$v(t) := \int_t^\omega q_2(s)\nabla s + \int_t^\omega F(s)\nabla s \leq z(t),$$

where the inequality is due to (3.4). Also

$$v^\nabla(t) = -q_2(t) - \frac{(z^\rho(t))^2}{p_1^\rho(t) + \nu(t) z^\rho(t)}.$$

Then since for each fixed t, the function $H(m) = \frac{m^2}{p_1^\rho(t)+\nu(t)m}$ is strictly increasing for $m \geq 0$, we have

$$v^\nabla(t) + q_2(t) + \frac{(v^\rho(t))^2}{p_1^\rho(t) + \nu(t)v^\rho(t)} \leq 0.$$

We thus have by Theorem 2.13 that

$$\left(p_1(t)x^\Delta\right)^\nabla + q_2(t)x = 0$$

is nonoscillatory on $[a, \omega)$, and then by Theorem 2.14 and (3.1) that $L_2 x = 0$ is nonoscillatory on $[a, \omega)$. □

Theorem 3.6. *Suppose the conditions of Theorem 3.5 hold with* (3.4) *replaced by*

$$\left| \int_t^\omega q_2(s)\nabla s \right| \leq \int_t^\omega q_1(s)\nabla s, \quad t \in [a, \omega), \tag{3.8}$$

and (3.5) *replaced by*

(3.9)

$$\exists\, M > m > 0 \quad \textit{such that} \quad m\nu(t) \leq p_1^\rho(t) \leq M\nu(t) \quad \textit{provided} \quad \nu(t) > 0.$$

If $q_1(t_k) > 0$ *for sufficiently large* t_k *and*

$$\liminf_{k\to\infty} \frac{q_2(t_k)}{q_1(t_k)} > -1 \tag{3.10}$$

for $\{t_k\}$ *as previously defined, then the same conclusion holds.*

Proof. Since (3.9) is a stronger condition then (3.5), we need only show that

$$\frac{(z^\rho(t))^2}{p_1^\rho(t) + \nu(t)z^\rho(t)} \geq \frac{(v^\rho(t))^2}{p_1^\rho(t) + \nu(t)v^\rho(t)}$$

for $t \in [T, \omega)$. However, we do not have $0 \leq v(t) \leq z(t)$ for $t \in [T, \omega)$ as before in the proof of Theorem 3.5, rather we have

$$|v(t)| \leq \left| \int_t^\omega q_2(s)\nabla s \right| + \int_t^\omega F(s)\nabla s \leq z(t)$$

for $t \in [T, \omega)$. The desired inequality is trivially true at left-dense points, and we have shown previously that it is true at points where $0 \leq v(t) \leq z(t)$, so we need only consider left-scattered points where $v(t) < 0$. At such points, an equivalent condition is

$$\begin{aligned}
p_1^\rho (z^\rho)^2 + \nu v^\rho (z^\rho)^2 &\geq p_1^\rho (v^\rho)^2 + \nu z^\rho (v^\rho)^2 \\
p_1^\rho \left[(z^\rho)^2 - (v^\rho)^2\right] &\geq \nu v^\rho z^\rho (v^\rho - z^\rho) \\
p_1^\rho (v^\rho + z^\rho) &\geq -\nu v^\rho z^\rho \\
\frac{p_1^\rho}{\nu} &\geq -\frac{v^\rho}{1 + \frac{v^\rho}{z^\rho}}.
\end{aligned}$$

Now provided that there is no sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \omega$ and

$$\liminf_{n\to\infty} \frac{v^\rho(t_n)}{z^\rho(t_n)} = -1$$

we then have

$$\lim_{t\to\infty} \frac{v^\rho(t)}{1+\frac{v^\rho(t)}{z^\rho(t)}} = 0$$

and then by (3.9) we have that the desired condition holds. We now assume that we have such a sequence $\{t_n\}$, and produce a contradiction. Then as $q_1(t) > 0$ for sufficiently large t, we have $z^\nabla(t) < 0$ for sufficiently large t. Then using L'Hôpital's rule for the time scale nabla case (Theorem 1.4) we get that

$$\begin{aligned} -1 = \lim_{n\to\infty} \frac{v(t_n)}{z(t_n)} &= \lim_{n\to\infty} \frac{v^\nabla(t_n)}{z^\nabla(t_n)} \\ &= \lim_{n\to\infty} \frac{q_2 + \frac{(z^\rho)^2}{p_1^\rho+\nu z^\rho}}{q_1 + \frac{(z^\rho)^2}{p_1^\rho+\nu z^\rho}} \\ &= \lim_{n\to\infty} \frac{q_2\left(p_1^\rho+\nu z^\rho\right)+\left(z^\rho\right)^2}{q_1\left(p_1^\rho+\nu z^\rho\right)+\left(z^\rho\right)^2} \\ &= \lim_{n\to\infty} \frac{a_n\frac{q_2}{q_1}+b_n}{a_n+b_n}, \end{aligned}$$

where

$$a_n = p_1^\rho(t_n)+\nu(t_n)z^\rho(t_n) \quad \text{and} \quad b_n = \left(z^\rho(t_n)\right)^2/q_1(t_n).$$

Now by (3.10) we may choose $0 < \varepsilon < 2$ such that $\frac{q_2(t_n)}{q_1(t_n)} > -1+\varepsilon$ for sufficiently large n. We may then choose $0<\delta<\varepsilon$ such that for sufficiently large n

$$\begin{aligned} \frac{a_n\frac{q_2(t_n)}{q_1(t_n)}+b_n}{a_n+b_n} &< -1+\delta \\ a_n\frac{q_2(t_n)}{q_1(t_n)}+b_n &< (-1+\delta)(a_n+b_n) \\ a_n(-1+\varepsilon)+b_n &< (-1+\delta)(a_n+b_n) \\ (\varepsilon-\delta)a_n &< (\delta-2)b_n \end{aligned}$$

This cannot be as $a_n > 0$ and $\varepsilon-\delta > 0$ while $b_n > 0$ and $\delta - 2 < 0$. □

4. Examples

Example 4.1. In this example we show that the q-difference equation

$$x^{\Delta\nabla} + \frac{1}{(q-1)t\log_q t}x = 0, \quad t \in q^{\mathbb{N}_0} \tag{4.1}$$

is oscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a constant. Let $a_0 := q^{k_0}$ for some fixed integer $k_0 > 0$. Let $\widetilde{q}(t) := \frac{1}{(q-1)t\log_q t}$ for $t \in \mathbb{T}$ and consider

$$\begin{aligned}\int_{a_0}^{\infty} \widetilde{q}(t)\nabla t &= \int_{q^{k_0}}^{\infty} \widetilde{q}(t)\nabla t \\ &= \sum_{j=k_0}^{\infty} \widetilde{q}(q^j)\nu(q^j) \\ &= \sum_{j=k_0}^{\infty} \frac{1}{(q-1)q^j j}(q^j - q^{j-1}) \\ &= \frac{1}{q}\sum_{j=k_0}^{\infty} \frac{1}{j} \\ &= +\infty.\end{aligned}$$

Then by Wintner's Theorem (Theorem 3.1), equation (4.1) is oscillatory on $\mathbb{T}$.

Example 4.2. Let $\mathbb{T} = \{t_k\}_{k=0}^{\infty} \bigcup \{1\}$ with $t_k = 1 - \left(\frac{1}{2}\right)^k$. Then for $k > 0$,

$$\nu(t_k) = \frac{1}{2^k}, \quad \text{and} \quad \mu(t_k) = \frac{1}{2^{k+1}}.$$

We claim that the dynamic equation

$$\left(-\frac{\ln(1-t)}{\ln 2} 2^{\frac{\ln(1-t)}{\ln 2}} x^{\Delta}(t)\right)^{\nabla} + 2^{-\frac{\ln(1-t)}{\ln 2}} x = 0 \tag{4.2}$$

is oscillatory on the time scale interval $[0, 1)$. Choose $a = t_j$ for a fixed integer $j > 0$, then

$$\begin{aligned}\int_{a}^{\omega} \frac{1}{p(t)}\Delta t &= \int_{t_j}^{1} \frac{1}{p(t)}\Delta t \\ &= \sum_{k=j}^{\infty} \frac{1}{p(t_k)}\mu(t_k) \\ &= \sum_{k=j}^{\infty} \frac{1}{-\frac{\ln(1-t_k)}{\ln 2} 2^{\frac{\ln(1-t_k)}{\ln 2}}}\mu(t_k) \\ &= \sum_{k=j}^{\infty} \frac{\ln 2}{-\ln((1/2)^k) 2^{\frac{\ln((1/2)^k)}{\ln 2}}} \frac{1}{2^{k+1}} \\ &= \sum_{k=j}^{\infty} \frac{\ln 2}{k\ln(2) 2^{\frac{-k\ln 2}{\ln 2}}} \frac{1}{2^{k+1}} \\ &= \sum_{k=j}^{\infty} \frac{1}{2k} = +\infty.\end{aligned}$$

We similarly have that

$$\int_a^\omega q(t)\nabla t = \sum_{k=j}^{\infty} 2^{-\frac{\ln((1/2)^k)}{\ln 2}} \nu(t_k)$$
$$= \sum_{k=j}^{\infty} 2^k \frac{1}{2^k} = +\infty$$

Then the Leighton–Wintner Theorem (Theorem 3.4) implies that (4.2) is oscillatory on $[0, 1)$.

Example 4.3. Consider the time scale $\mathbb{T} = \cup_{i=1}^{\infty}\{c_i\} \bigcup \cup_{i=1}^{\infty}\{d_i\}$, where $c_1 < d_1 < c_2 < d_2 < \ldots$ and $d_i - c_i = \left(\frac{1}{i}\right)^2$ and $c_{i+1} - d_i = i$ and $c_1 = 1$. Suppose $p(t) \equiv 1$ and $q(c_i) = \left(\frac{1}{i}\right)^2$ and $q(d_i) = \left(\frac{1}{i}\right)^5$. We then have that

$$\int_1^\infty \frac{1}{p(t)}\Delta t = \int_1^\infty 1 \; \Delta t = \infty$$

and

$$\int_1^\infty q(t)\nabla t = \sum_{i=1}^{\infty} \left[\left(\frac{1}{i}\right)^7 + \frac{i}{(i+1)^2} \right] = +\infty$$

Thus the Leighton–Wintner theorem guarantees that (1.1) is oscillatory on $[1, \infty)$. However note that

$$\int_1^\infty q(t)\Delta t = \sum_{i=1}^{\infty} \left[\left(\frac{1}{i}\right)^4 + \left(\frac{1}{i}\right)^4 \right] < +\infty,$$

which is not the assumption needed in the analogue of the theorem for the equation $\left(px^\Delta\right)^\Delta + qx^\sigma = 0$, found in [2, Theorem 4.64]. We leave it to the interested reader to show that this last equation is indeed nonoscillatory on $[1, \infty)$.

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