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The Henstock–Kurzweil Delta Integral on Unbounded Time Scales

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Abstract

In this paper we will investigate the Henstock–Kurzweil Δ-integral on unbounded time scale intervals. First we will define a time scale and give a brief overview of time scale calculus. Then we will give theorems relating to integrability, followed by examples of functions on various time scales with specific focus on the time scale $q^{\mathbb{N}_0}$. We will prove every improper Δ-integrable function is $HK\Delta$-integrable and that the Monotone and Dominated Convergence Theorems hold for $HK\Delta$-integrable functions on unbounded time scale intervals.

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1 Introduction

For a good book on the Henstock–Kurzweil integral see Peng–Yee [7]. Peterson and Thompson [8] defined and studied the Henstock–Kurzweil Δ-integral on bounded time scale intervals. In this paper, we shall study the Henstock–Kurzweil Δ-integral ($HK\Delta$-integral) on unbounded time scales. This $HK\Delta$-integral which generalizes the improper Δ-integral (improper Riemann Δ-integral), is useful in the study of dynamic equations. In [8] it is shown that there are highly oscillatory functions that are not Δ-integrable on a time scale, but are

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Henstock–Kurzweil Δ-integral. We begin with an overview of time scales and proceed to define the $HK\Delta$-integral on unbounded intervals. We then establish properties and offer theorems pertaining to the integral. We first give some basic definitions (for an introduction to the time scale calculus see the books [1], [2], and [3]).

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ equipped with the topology inherited from the standard topology on $\mathbb{R}$. The *forward jump operator* $\sigma(t)$ of $t \in \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$$

where $\inf \emptyset = \sup\{\mathbb{T}\}$ and the *backward jump operator* $\rho(t)$ of $t \in \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$$

where $\sup \emptyset = \inf\{\mathbb{T}\}$. The *(forward) graininess* function $\mu(t)$ of $t \in \mathbb{T}$ is defined by

$$\mu(t) = \sigma(t) - t$$

the *backward graininess* function $\nu(t)$ of $t \in \mathbb{T}$ is defined by

$$\nu(t) = t - \rho(t).$$

A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$. Likewise, a point $t \in \mathbb{T}$ is *left-scattered* if $t > \rho(t)$. A point $t \in \mathbb{T}$ is *right-dense* if $\sigma(t) = t$ and $t < \sup\{\mathbb{T}\}$. Likewise, a point $t \in \mathbb{T}$ is *left-dense* if $\rho(t) = t$ and $t > \inf\{\mathbb{T}\}$. For some simple examples of these concepts (for many more examples see [1]) note that $\mathbb{T} := \mathbb{R}$ is a time scale with $\sigma(t) = \rho(t) = t$ and $\mu(t) = \nu(t) = 0$ for all $t \in \mathbb{T}$. In this case, t is right-dense for all $t \in \mathbb{T}$. $\mathbb{T} := \mathbb{Z}$ is a time scale with $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = \nu(t) = 1$ for all $t \in \mathbb{T}$. In this case, t is right-scattered for all $t \in \mathbb{T}$. $\mathbb{T} := \mathbb{N}^p := \{n^p : n \in \mathbb{N}\}$ is a time scale with, for $p \in \mathbb{N}$, $\sigma(t) = \sum_{j=0}^{p} \binom{p}{j} t^{1-\frac{j}{p}}$, $\rho(t) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} t^{1-\frac{j}{p}}$, $\mu(t) = \sum_{j=1}^{p} \binom{p}{j} t^{1-\frac{j}{p}}$, and $\nu(t) = \sum_{j=1}^{p} (-1)^j \binom{p}{j} t^{1-\frac{j}{p}}$ for all $t \in \mathbb{T}$. In this case, t is right-scattered for all $t \in \mathbb{T}$. $\mathbb{T} := q^{\mathbb{N}} := \{q^n : n \in \mathbb{N}, q > 1\}$ is a time scale with $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$, $\mu(t) = (q-1)t$, and $\nu(t) = (1 - \frac{1}{q})t$ for all $t \in \mathbb{T}$. In this case, t is right-scattered for all $t \in \mathbb{T}$. $\mathbb{T} := \frac{1}{m}\mathbb{Z}^+ := \{\frac{n}{m} : n \in \mathbb{Z}^+\}$ is a time scale for $m > 0$ with $\sigma(t) = t + \frac{1}{m}$, $\rho(t) = t - \frac{1}{m}$, and $\mu(t) = \nu(t) = \frac{1}{m}$ for all $t \in \mathbb{T}$. In this case, t is right-scattered for all $t \in \mathbb{T}$. $\mathbb{T} := \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1]$ is a time scale with $\sigma(t) = t$ for $t \in [2k, 2k+1)$ and $\sigma(t) = t + 1$ for $t = 2k+1$; $\rho(t) = t$ for $t \in (2k, 2k+1]$ and $\rho(t) = t - 1$ for $t = 2k$; $\mu(t) = 0$ for $t \in [2k, 2k+1)$ and $\mu(t) = 1$ for $t = 2k+1$; and $\nu(t) = 0$ for $t \in (2k, 2k+1]$ and $\nu(t) = 1$ for $t = 2k$.

Throughout this paper, we use the notation $[a, b]_{\mathbb{T}}$ to denote a *time scale interval* where $[a, b]_{\mathbb{T}} := \mathbb{T} \bigcap [a, b]$, where $a, b \in \mathbb{T}$. A time scale $\mathbb{T}$ is said to be *isolated* if t is right-scattered and left-scattered for all $t \in \mathbb{T}$, where $t \neq \sup\{\mathbb{T}\}$ and $t \neq \inf\{\mathbb{T}\}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense $t \in \mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Remark 1.1. Any function defined on an isolated time scale is rd-continuous. (This is vacuously true because $\mathbb{T}$ has no right-dense or left-dense elements in this case.) If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is rd-continuous iff it is continuous.

Let $\mathbb{T}^\kappa := \mathbb{T} - m$ if $\mathbb{T}$ has a left-scattered maximum m and let $\mathbb{T}^\kappa := \mathbb{T}$ if $\mathbb{T}$ has no right-scattered maximum.

Definition 1.1. Assume $f : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^\kappa$, then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)\big| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the Δ-*derivative* of f at t.

Remark 1.2. If f is continuous at t, it can be shown for right-scattered $t \in \mathbb{T}$

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

and for right-dense $t \in \mathbb{T}$

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Definition 1.2. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(t) = f(t)$$

holds for all $t \in \mathbb{T}^\kappa$.

It is known ([1, Section 1.4]) that if $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then f has an antiderivative F and

$$\int_a^b f(s)\Delta s = F(b) - F(a).$$

2 Integrability and Power Functions

In this section, we will discuss the concepts of integrability and power functions on time scales. We will give examples and properties of both, thereby forming a basis for the material discussed later in the paper.

Let $a \in \mathbb{T}$, and assume f is rd-continuous on $[a, \infty)_\mathbb{T}$, then the improper Δ-integral, denoted by $\int_a^\infty f(t)\Delta t$, is defined in the normal way by

$$\int_a^\infty f(t)\Delta t := \lim_{b \to \infty} \int_a^b f(t)\Delta t$$

provided this limit exists. For an isolated time scale $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$ which is unbounded above the improper integral exists provided

$$\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(t_k)(t_{k+1} - t_k)$$

converges [2].

Definition 2.1. A *power function* $t^{[p]}$ on a time scale is a function f satisfying the power rule $(t^{[p]})^{\Delta} = pt^{[p-1]}$, with $t^{[0]} = 1$.

Example 1: A power function for $\mathbb{T} = \mathbb{R}$ is $t^{[p]} = t^p$. This is simply because $t^{[0]} = t^0 = 1$ and

$$(t^{[p]})^{\Delta} = \frac{d}{dt}(t^p) = pt^{p-1} = pt^{[p-1]}$$

for values of t and p such that the above is defined.
Example 2: A power function for $\mathbb{T} = \mathbb{Z}$ (see [4]) is given by $t^{[p]} = \frac{\Gamma(t+1)}{\Gamma(t-p+1)}$, where $\Gamma(t)$ is the gamma function, for those values of $t \in \mathbb{Z}$ and p for which the right hand side of this last equation is defined.

We next define some power functions on $q^{\mathbb{N}_0}$.

Definition 2.2. Let $Q_0(q) := 1$ and assume $Q_\alpha(q) \neq 0$ is arbitrary for $\alpha \in (0,1)$.We define $Q_{n+\alpha}(q)$ by

$$Q_{n+\alpha}(q) = Q_\alpha(q)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\prod_{k=1}^{n}\frac{q^{k+\alpha}-1}{q-1} \tag{2.1}$$

for $n \in \mathbb{N}$ and $\alpha \in [0,1)$. For $\alpha \in (-1,0)$, $n \in \mathbb{N}_0$ let

$$Q_{\alpha-n}(q) = \frac{\Gamma(\alpha+1)(q-1)^n}{\Gamma(\alpha-n+1)\prod_{k=0}^{n-1}(q^{\alpha-k}-1)}Q_\alpha(q) \tag{2.2}$$

and finally let $Q_{-1}(q) \neq 0$ be arbitrary

$$Q_{-n-1}(q) := \frac{n!(q-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(q^k-1)}Q_{-1}(q). \tag{2.3}$$

for $n \in \mathbb{N}$.

It is straight forward to prove the following two lems and so the proofs are omitted.

Lemma 2.1. *For each $\alpha \in [0,1)$ the recursion relation*

$$Q_{n+\alpha+1}(q) = \frac{q^{n+\alpha+1}-1}{(n+\alpha+1)(q-1)}Q_{n+\alpha}(q) \tag{2.4}$$

holds for $n \in \mathbb{N}_0$*. For* $\alpha \in (-1, 0)$ *the recursion relation*

$$Q_{\alpha-n-1}(q) = \frac{(\alpha - n)(q-1)}{q^{\alpha-n} - 1} Q_{\alpha-n}(q) \tag{2.5}$$

holds for $n \in \mathbb{N}$*. Finally the recursion relation*

$$Q_{-n-1}(q) = \frac{nq^n(q-1)}{q^n - 1} Q_{-n}(q) \tag{2.6}$$

holds for $n \in \mathbb{N}$*.*

Lemma 2.2. *For* $n \in \mathbb{N}_0$

$$Q_n(q) = \frac{1}{n!} \prod_{k=1}^{n} \frac{q^k - 1}{q - 1} \tag{2.7}$$

and for $n \in \mathbb{N}$

$$Q_{-n}(q) = \frac{q^{\frac{n(n-1)}{2}}}{Q_{n-1}(q)} Q_{-1}(q). \tag{2.8}$$

Furthermore $Q_{-n}(q)$ *satisfies the recurtion relation*

$$Q_{-(n+1)}(q) = q^n \frac{Q_{n-1}(q)}{Q_n(q)} Q_{-n}(q) = \frac{-n(q-1)}{q^{-n} - 1} Q_{-n}(q). \tag{2.9}$$

Theorem 2.3. *Power functions on* $\mathbb{T} = q^{\mathbb{N}}$ *are given by*

$$t^{[p]} = \frac{t^p}{Q_p(q)}.$$

Proof. We will just prove this result for $p \geq 0$ as the other cases are similar. First note that $t^{[0]} = 1$ so $(t^{[1]})^\Delta = \frac{1}{Q_1(q)} = 1 = t^{[0]}$. Now asume $p > 0$. Then $p = n + \alpha$ for some $n \in \mathbb{N}_0$ and $\alpha \in [0, 1)$. Consider

$$\begin{aligned}
(t^{[p]})^\Delta &= (t^{[n+\alpha]})^\Delta = \left(\frac{t^{n+\alpha}}{Q_{n+\alpha}(q)} \right)^\Delta \\
&= \frac{\frac{(\sigma(t))^{n+\alpha}}{Q_{n+\alpha}(q)} - \frac{t^{n+\alpha}}{Q_{n+\alpha}(q)}}{\mu(t)} \\
&= \frac{(qt)^{n+\alpha} - t^{n+\alpha}}{Q_{n+\alpha}(q)(q-1)t} \\
&= \frac{(q^{n+\alpha} - 1)t^{n+\alpha-1}}{(q-1)Q_{n+\alpha}(q)} \\
&= (n+\alpha) \frac{t^{n+\alpha-1}}{Q_{n+\alpha-1}(q)}, \quad \text{by} \quad (2.4) \\
&= pt^{[p-1]}.
\end{aligned}$$

Hence the result holds for $n \geq 0$. ☐

We will use the following example in Section 3.
Example 3: Let $\mathbb{T} = q^{\mathbb{N}}$. Then

$$\int_1^\infty \frac{\Delta t}{t^p} = \frac{q^{p-1}(q-1)}{q^{p-1}-1}$$

where $p > 1$ and $q > 1$.

Proof. First we consider

$$\begin{aligned}
\int_1^\infty \frac{\Delta t}{t^p} &= \int_1^\infty t^{[-p]} Q_{-p}(q) \Delta t \\
&= Q_{-p}(q) \int_1^\infty t^{[-p]} \Delta t \\
&= Q_{-p}(q) \frac{t^{[1-p]}}{1-p}\Bigg|_1^\infty \\
&= Q_{-p}(q) \frac{t^{1-p}}{(1-p)Q_{1-p}(q)}\Bigg|_1^\infty \\
&= \frac{Q_{-p}(q)}{(p-1)Q_{1-p}(q)}.
\end{aligned}$$

From (2.6),

$$\frac{Q_{-p}(q)}{Q_{1-p}(q)} = \frac{(p-1)q^{p-1}(q-1)}{q^{p-1}-1}$$

implying

$$\int_1^\infty \frac{\Delta t}{t^p} = \frac{q^{p-1}(q-1)}{q^{p-1}-1}.$$

□

3 The $HK\Delta$-Integral

In this section we will introduce the Henstock–Kurzweil Δ-integral as defined on unbounded time scales. After defining this integral, we will compare it to the improper Δ-integral and provide examples and proofs. A simple result known as Henstock's Lemma will be used in this section as well as the following.

Definition 3.1. A *partition* P on $[a,b]_{\mathbb{T}}$ is defined as $P = \{t_i \in \mathbb{T}; i = 0, \cdots, n\}$ where $\{a = t_0 \le \xi_1 \le t_1 \le \xi_2 \le t_2 \le \cdots \le t_n = b\}$, and $t_{i-1} < t_i$ for $i = 1, \cdots, n$. We denote $\mathcal{P}_{[a,b]}$ as the set of all partitions on $[a,b]_{\mathbb{T}}$. We call each t_i an endpoint and each ξ_i a tag-point. It is necessary that $\xi_i \in \mathbb{T}$ for $i = 1, \cdots, n$.

Definition 3.2. We say that $\delta(t) = (\delta_L(t), \delta_R(t))$ is a Δ-*gauge* for $[a,\infty]_{\mathbb{T}}$, where we will define $[a,\infty]_{\mathbb{T}} := [a,\infty)_{\mathbb{T}} \bigcup \{\infty\}$, provided

$$\begin{aligned} \delta_L(t) > 0, &\quad t \in (a,\infty]_{\mathbb{T}} \\ \delta_R(t) > 0, &\quad t \in [a,\infty)_{\mathbb{T}} \\ \delta_L(a) \geq 0 & \\ \delta_R(t) \geq \mu(t), &\quad t \in [a,\infty)_{\mathbb{T}} \\ \delta_L(\infty) = B. & \end{aligned}$$

where $B \in \mathbb{R}^+$. Since for a Δ-gauge, δ, we always assume $\delta_L(a) \geq 0$, we will sometimes not even point this out.

Definition 3.3. If δ is a Δ-gauge on $[a,\infty]_{\mathbb{T}}$, then $P \in \mathcal{P}_{[a,\infty]}$ is a δ-*fine partition* provided

$$[\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)] \supset [t_{i-1}, t_i]$$

for $i = 1, ..., n$ and $t_n > B$.

Note that if $\mathbb{T}$ is an isolated time scale, then if we choose our Δ-gauge to be

$$\begin{aligned} \delta_R(t) &= \mu(t) \\ \delta_L(t) &= \frac{1}{2}\nu(t), t < \infty \\ \delta_L(\infty) &= B > 0 \end{aligned}$$

where B is dependent on the function and the time scale, then we force any δ-fine partition to follow certain rules. Assume $\xi_i \neq t_0$. Because the first endpoint in our partition must be t_0 and $\delta_L(\xi_1) = \frac{1}{2}\nu(\xi_1)$, we get $[\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)] \not\supset [t_{i-1}, t_i]$, a contradiction. So, we must set our first tag-point $\xi_1 = t_0$. From there on, the farthest away that any tag-point ξ_i can be from ξ_{i-1} is $\mu(\xi_{i-1})$, which means that each successive tag-point has to occur at the next endpoint, starting with $\xi_1 = t_0$. This choice of a Δ-gauge assures that for any δ-fine partition P, $P = [t_0, t_n]_{\mathbb{T}}$, with $\xi_i = t_{i-1}$ for $i = 1, ..., n$.

Definition 3.4. We say $f : [a,\infty]_{\mathbb{T}} \to \mathbb{R}$ is *Henstock–Kurzweil* Δ-*integrable* on $[a,\infty)_{\mathbb{T}}$ provided there is a number I such that given $\varepsilon > 0$ there exists a Δ-gauge, δ, of $[a,\infty]_{\mathbb{T}}$ such that

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon$$

for every δ-fine partition of $[a,\infty]_{\mathbb{T}}$, and we write

$$I = HK \int_a^\infty f(t)\Delta t.$$

The following are demonstrations of how to prove $HK\Delta$-integrability.

Example 4: Let $\mathbb{T} = q^{\mathbb{N}}$. Then

$$HK\int_1^\infty \frac{\Delta t}{t^p} = \frac{q^{p-1}(q-1)}{q^{p-1}-1}$$

where $p > 1$ and $q > 1$.

Proof. Given $I > \varepsilon > 0$, we consider

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| \tag{3.1}$$

where

$$f(\xi_i) = \frac{1}{\xi_i^p} \tag{3.2}$$

and

$$I = \frac{q^{p-1}(q-1)}{q^{p-1}-1}.$$

(Note that $I > 0$ because $p > 1$ and $q > 1$). First we define our Δ-gauge to be

$$\delta_R(t) = \mu(t)$$
$$\delta_L(t) = \frac{1}{2}\nu(t), \quad t < \infty$$
$$\delta_L(\infty) = \Big(\frac{I}{\varepsilon}\Big)^{\frac{1}{p-1}}.$$

Observe that

$$\sum_{i=1}^{n} I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big) = I\Big(\frac{1}{t_0^{p-1}} - \frac{1}{t_n^{p-1}}\Big).$$

Since $t_0 = 1$ in this example, we get

$$I\Big(1 - \frac{1}{t_n^{p-1}}\Big) = \sum_{i=1}^{n} I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big)$$

which we rewrite as

$$I = \frac{I}{t_n^{p-1}} + \sum_{i=1}^{n} I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big) \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we get

$$\left|\frac{I}{t_n^{p-1}} + \sum_{i=1}^{n} I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big) - \sum_{i=1}^{n} \frac{1}{\xi_i^p}(t_i - t_{i-1})\right| \tag{3.4}$$

$$= \left|\frac{I}{t_n^{p-1}} + \sum_{i=1}^{n} \Big[I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big) - \frac{1}{\xi_i^p}(t_i - t_{i-1})\Big]\right|$$

$$\leq \left|\frac{I}{t_n^{p-1}}\right| + \sum_{i=1}^{n} \left|I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big) - \frac{1}{\xi_i^p}(t_i - t_{i-1})\right|$$

$$= \left|\frac{I}{t_n^{p-1}}\right| + \sum_{i=1}^{n} \left(\left|I\Big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\Big)\right| \left|1 - \frac{t_i - t_{i-1}}{I\xi_i^p\big(\frac{1}{t_{i-1}^{p-1}} - \frac{1}{t_i^{p-1}}\big)}\right|\right).$$

Now by the definition of our Δ-gauge we have forced that $t_i = q^i$ and $\xi_i = t_{i-1} = q^{i-1}$. We substitute these values into the right part of our summand to get

$$\left|1 - \frac{q^i - q^{i-1}}{Iq^{ip}\big(\frac{1}{q^{(i-1)(p-1)}} - \frac{1}{q^{i(p-1)}}\big)}\right| = \left|1 - \frac{q^{p-1}(q-1)}{I(q^{p-1}-1)}\right| = \left|1 - \frac{I}{I}\right| = |1-1| = 0.$$

So our whole summand becomes 0, leaving us with (3.4) reducing to only $\left|\frac{I}{t_n^{p-1}}\right|$. Pick $n \geq N$ where

$$t_N > \Big(\frac{I}{\varepsilon}\Big)^{\frac{1}{p-1}}$$

i.e.,

$$N > \frac{\log_q(\frac{I}{\varepsilon})}{(p-1)}.$$

Then we have

$$\left|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| < \varepsilon.$$

□

Example 5: For $\mathbb{T} = \frac{1}{m}\mathbb{Z}^+$,

$$HK\int_{\frac{1}{m}}^{\infty} \frac{\sin^{2b+1}(t)}{t}\Delta t = \sum_{i=1}^{\infty} \frac{\sin^{2b+1}(\frac{i}{m})}{i}$$

$$= \frac{(2b)!}{2^{2b+1}b!^2}\Big(\pi - \frac{1}{m}\Big) + \sum_{j=0}^{b}(-1)^j \binom{2b+1}{j+b+1}\Big(\pi\Big\lfloor\frac{2j+1}{2m\pi}\Big\rfloor - \frac{j}{m}\Big) \tag{3.5}$$

where $b \in \mathbb{N}$. The proof that the above series converges to the given value is left as an exercise in elementary complex analysis.

Proof. If we let our Δ-gauge be defined as

$$\delta_R(t) = \mu(t)$$
$$\delta_L(t) = \frac{1}{2}\nu(t), \quad t < \infty$$
$$\delta_L(\infty) = \frac{N}{m}$$

where N is defined later in the example. By the definition of our Δ-gauge, $t_i = \frac{i+1}{m}$ and $\xi_i = t_{i-1} = \frac{i}{m}$. Note that $t_i - t_{i-1} = \frac{i+1}{m} - \frac{i}{m} = \frac{1}{m}$. So given $\varepsilon > 0$, we consider

$$\begin{aligned} \left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| &= \left| I - \sum_{i=1}^{n} \frac{\sin^{2b+1}(\frac{i}{m})}{\frac{i}{m}} (\frac{1}{m}) \right| \\ &= \left| \sum_{i=1}^{\infty} \frac{sin^{2b+1}(\frac{i}{m})}{i} - \sum_{i=1}^{n} \frac{sin^{2b+1}(\frac{i}{m})}{i} \right| \\ &= \left| \sum_{i=n+1}^{\infty} \frac{sin^{2b+1}(\frac{i}{m})}{i} \right|. \end{aligned}$$

By the definition of convergent series, there is an N: for all $n \geq N$,

$$\left| \sum_{i=n+1}^{\infty} \frac{sin^{2b+1}(\frac{i}{m})}{i} \right| < \varepsilon.$$

□

Note that if we informally take the limit of both sides of (3.5) as $m \to \infty$ we get the known result

$$\int_0^{\infty} \frac{\sin^{2b+1}(x)}{x} dx = \frac{(2b)!}{2^{2b+1} b!^2} \pi$$

for $b \in \mathbb{N}$.

Theorem 3.1. *If $\mathbb{T}$ is an isolated time scale which is unbounded above and $f : \mathbb{T} \to \mathbb{R}$ is improper Δ-integrable on $[t_0, \infty)_{\mathbb{T}}$, then f is $HK\Delta$-integrable on $[t_0, \infty)_{\mathbb{T}}$ with*

$$HK \int_{t_0}^{\infty} f(t)\Delta t = \int_{t_0}^{\infty} f(t)\Delta t =: I$$

Proof. Let $\mathbb{T} = \{t_0, t_1, t_2, ...\}$, where $t_{i-1} < t_i$, $i \in \mathbb{N}$. We define our Δ-gauge, $\delta = (\delta_L, \delta_R)$, by

$$\delta_R(t) = \mu(t)$$

$$\delta_L(t) = \frac{1}{2}\nu(t), \quad t < \infty$$
$$\delta_L(\infty) = t_N.$$

where N will be chosen later. Then if P is a δ-fine partition of $[t_0, \infty]_{\mathbb{T}}$ we must have $t_i = \sigma(t_{i-1})$, $\xi_i = t_{i-1}$, for $i = 1, 2, 3, \cdots, n$, and $t_n > t_N$. Let

$$F(t) := F(t_k) = \sum_{i=1}^{k-1} f(t_i)\mu(t_i),$$

then $F^{\Delta}(t) = f(t)$ for every $t \in \mathbb{T}$. Let $C := I + F(t_0)$, where $I := \int_{t_0}^{\infty} f(t)\Delta t$, then

$$I = C - F(t_n) + \sum_{i=1}^{n} \left[F(t_i) - F(t_{i-1})\right].$$

Now given $\varepsilon > 0$, we consider

$$\begin{aligned}
\left|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| &= \left|C - F(t_n) + \sum_{i=1}^{n}\left[F(t_i) - F(t_{i-1})\right] - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| \\
&= \left|C - F(t_n) + \sum_{i=1}^{n}\left[F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})\right]\right| \\
&\le |C - F(t_n)| + \sum_{i=1}^{n} \left|F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})\right|.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{i=1}^{n} \left|F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})\right| &= |t_i - t_{i-1}| \left|\frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}} - f(\xi_i)\right| \\
&= \mu(t_{i-1}) \left|\frac{F(\sigma(t_{i-1})) - F(t_{i-1})}{\mu(t_{i-1})} - f(t_{i-1})\right| \\
&= \mu(t_{i-1}) \left|F^{\Delta}(t_{i-1}) - f(t_{i-1})\right| \\
&= \mu(t_{i-1}) \left|f(t_{i-1}) - f(t_{i-1})\right| = 0.
\end{aligned}$$

Hence

$$\left|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| \le |C - F(t_n)| = |I + F(t_0) - F(t_n)| = |I - (F(t_n) - F(t_0))|.$$

Using

$$\int_{t_0}^{t_n} f(t)\Delta t = F(t_n) - F(t_0), \quad \int_{t_0}^{\infty} f(t)\Delta t = I$$

we have

$$\begin{aligned}|I-(F(t_n)-F(t_0))| &= \left|\int_{t_0}^{\infty} f(t)\Delta t - \int_{t_0}^{t_n} f(t)\Delta t\right| \\ &= \left|\int_{t_n}^{\infty} f(t)\Delta t\right|.\end{aligned}$$

Because f is improper Δ-integrable on $[t_0,\infty)_{\mathbb{T}}$, we are assured that there is a positive integer N such that

$$\int_{t_n}^{\infty} f(t)\Delta t < \varepsilon$$

for all $n \geq N$. Therefore f is $HK\Delta$-integrable on $[t_0,\infty)_{\mathbb{T}}$ with

$$HK\int_{t_0}^{\infty} f(t)\Delta t = \int_{t_0}^{\infty} f(t)\Delta t.$$

□

Example 6: We show that if $\mathbb{T}=\mathbb{N}^p$, $p\in\mathbb{N}$, and $m>1$, then

$$HK\int_1^{\infty}\frac{\Delta t}{t^m} = \sum_{j=1}^{p}\binom{p}{j}\zeta(j+(m-1)p),$$

where $\zeta(t)$ is the Riemann zeta function.

Proof. Since $\mathbb{N}^p$ is isolated with $\sup\mathbb{N}^p=\infty$, it suffices by Theorem 3.1 to show that the improper Δ-integral

$$\int_1^{\infty}\frac{\Delta t}{t^m} = \sum_{j=1}^{p}\binom{p}{j}\zeta(j+(m-1)p).$$

Let $t_k = k^p$, $k\in\mathbb{N}$ and consider

$$\begin{aligned}\int_1^{\infty}\frac{\Delta t}{t^m} &= \sum_{k=1}^{\infty}\frac{\mu(t_k)}{t_k^m} = \sum_{k=1}^{\infty}\frac{(k+1)^p-k^p}{k^{pm}m} = \sum_{k=1}^{\infty}\frac{1}{k^{mp}}\sum_{j=1}^{p}\binom{p}{j}(k^p)^{1-\frac{j}{p}} \\ &= \sum_{j=1}^{p}\binom{p}{j}\sum_{k=1}^{\infty}\frac{1}{k^{j+(m-1)p}} = \sum_{j=1}^{p}\binom{p}{j}\zeta(j+(m-1)p)\end{aligned}$$

(since $j+(m-1)p>1$, we know ζ is defined. □

Example 7: If $\mathbb{T}=\mathbb{N}^2$ it can be shown, similar to the previous example, that

$$HK\int_1^{\infty}\frac{(-1)^{1+\sqrt{t}}}{t}\Delta t = 2\ln(2)+\frac{\pi^2}{12}.$$

Example 8: If $\mathbb{T} = q^{\mathbb{N}}$ and $p > 1$, we can also show

$$HK \int_q^\infty \frac{\Delta t}{t \log_q^p t} = (q-1)\zeta(p).$$

Lemma 3.2 (Henstock's Lemma). *Let $HK \int_a^b f(t)\Delta t = I$, and let δ be a Δ-gauge such that for a fixed $\varepsilon > 0$, for any δ-fine partition $P \in \mathcal{P}_{[a,b]}$:*

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon.$$

Then for any subset of tagged intervals, $\mathcal{E}$, of P.

$$\left| \sum_{\mathcal{E}} \left(HK \int_u^v f(t)\Delta t - f(\xi)(v-u) \right) \right| \le \varepsilon.$$

Proof. Let δ be a Δ-gauge such that for any δ-fine partition P,

$$\left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| = \left| \sum_{i=1}^{n} \left(HK \int_{t_{i-1}}^{t_i} f(t)\Delta t - f(\xi_i)(t_i - t_{i-1}) \right) \right| < \varepsilon.$$

For any $\mathcal{E} \subseteq P$, let $\Phi(\mathcal{E}) = \sum_{\mathcal{E}} \left(HK \int_u^v f(t)\Delta t - f(\xi)(v-u) \right)$. Given some $\mathcal{E} \subseteq P$, $\Phi(\mathcal{E}) = \Phi(P) - \Phi(P \setminus \mathcal{E})$. Let n be the number of tagged intervals in $P \setminus \mathcal{E}$. Then for each tagged interval $([u,v],\xi) \in P \setminus \mathcal{E}$, there exists a refinement δ^i of δ such that for any δ^i-fine partition $\mathcal{P}^i$ of $[u,v]$, $|\Phi(\mathcal{P}^i)| < \frac{\varepsilon'}{n}$. If we let $\mathcal{P}' = \mathcal{E} \cup \mathcal{P}^1 \cup \mathcal{P}^2 \cup ... \cup \mathcal{P}^n$ then,

$$|\Phi(\mathcal{E})| \le |\Phi(P')| + |\Phi(P' \setminus \mathcal{E})| \le \varepsilon + \sum_{i=1}^{n} \frac{\varepsilon'}{n} = \varepsilon + \varepsilon'.$$

This is true for any arbitrary $\varepsilon' > 0$. Hence, we may conclude that $|\Phi(\mathcal{E})| \le \varepsilon$ as desired.

□

Lemma 3.3. *As in [2, Lemma 5.7] we denote by $\mathcal{P}_\gamma(a,b)$ the set of all partitions of $[a,b]$ such that each partition $P \in \mathcal{P}_\gamma(a,b)$ given by $\{a = t_0 < t_1 < ... < t_n = b\}$ has the property that for $\gamma > 0$ and $i = 1, ..., n$ either*

$$t_i - t_{i-1} \le \gamma$$

or

$$t_i - t_{i-1} > \gamma \quad \text{and} \quad \rho(t_i) = t_{i-1}.$$

Theorem 3.4. *Let f be Riemann Δ-integrable on any finite interval $[a,b]_{\mathbb{T}}$ for $a < b < \infty$, $b \in \mathbb{T}$, and let $\int_a^\infty f(t)\Delta t = I$. Then f is $HK\Delta$-integrable on $[a,\infty)_{\mathbb{T}}$ and $HK \int_a^\infty f(t)\Delta t = I$.*

Proof. This proof is formatted after a proof of a similar fact on a finite interval with a singularity at one endpoint in [5, Theorem 4] and will use Henstock's Lemma. Fix $\varepsilon > 0$. Let $\{c_m\}_{m=0}^{\infty}$ be a strictly increasing sequence of numbers in $\mathbb{T}$ with $c_0 = a$ and

$$\lim_{m\to\infty} c_m = \infty.$$

Since f is Riemann Δ-integrable on $[a, c_2]$, there exists a number γ_1 such that $0 < \gamma_1 \leq min\{c_1 - c_0, c_2 - c_1\}$ and

$$\left|\int_{c_0}^{c_2} f(t)\Delta t - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| < \frac{\varepsilon}{2^2}$$

for any partition $P \in \mathcal{P}_{\gamma_1}(c_0, c_2)$. For $m = 2, 3, 4, ...$ let γ_m satisfy $0 < \gamma_m \leq c_{m+1} - c_m$, $\gamma_m \leq \gamma_{m-1}$, and

$$\left|\int_{c_{m-2}}^{c_{m+1}} f(t)\Delta t - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\right| < \frac{\varepsilon}{2^{m+1}}$$

for any partition $P \in \mathcal{P}_{\gamma_m}(c_{m-2}, c_{m+1})$. Now define a Δ-gauge, $\delta = (\delta_L, \delta_R)$, on $[a, \infty]_{\mathbb{T}}$ by

$$\delta_R(t) = \begin{cases} \frac{\gamma_m}{2}, & t \in [c_{m-1}, c_m),\ \ t \text{ right-dense} \\ \mu(t), & t \quad \text{right-scattered} \end{cases}$$

and

$$\delta_L(t) = \begin{cases} \frac{\gamma_m}{2}, & t \in [c_{m-1}, c_m) \\ A, & t = \infty, \end{cases}$$

where A is defined such that for any $A_1 > A$ and $A_2 > A$,

$$\left|\int_{A_1}^{A_2} f(t)\Delta t\right| < \frac{\varepsilon}{4}$$

and for any $c > A$,

$$\left|I - \int_{a}^{c} f(t)\Delta t\right| < \frac{\varepsilon}{4}.$$

Let P be a δ-fine partition of $[a, \infty]_{\mathbb{T}}$. For $m = 1, 2, 3, ...$ let N_m be the set of integers i for which $\xi_i \in [c_{m-1}, c_m)$ and $2 \leq i \leq n$, and then consider two cases. In the first case, assume $t_i - t_{i-1} \leq \gamma_m$. Then $m \geq 1$ and $i \in N_m$ implies

$$t_{i-1} \geq t_{i-1} - t_i + \xi_i \geq -\gamma_m + \xi_i \geq -\gamma_m + c_{m-1} \geq c_{m-2}.$$

Hence if $m \geq 2$, $i \in N_m$, and we set $\gamma_0 = c_1 - c_0$, then

$$c_{m-2} \leq t_{i-1} < t_i \leq t_i - t_{i-1} + \xi_i \leq \gamma_m + \xi_i < c_{m+1};$$

while if $i \in N_1$, then

$$c_0 \leq t_{i-1} < t_i < c_2.$$

In the second case consider $t_i - t_{i-1} > \gamma_m$. Again consider $m \geq 1$, $i \in N_m$, and $2 \leq i \leq n$. By the definition (Definition 3.3) of $P \in \mathcal{P}_{\gamma_m}(a,b)$ we must have that $\rho(t_i) = t_{i-1}$. In this case, assume $t_i > c_m$. This implies that since $\xi_i \in [t_{i-1}, t_i]$, and there are no points between t_{i-1} and t_i, then $\xi_i \geq c_m$, a contradiction. Hence

$$t_i < c_{m+1}.$$

Now assume $m \geq 2$ and $i \in N_m$. If we set $\gamma_0 = c_1 - c_0$ and assume also $t_{i-1} < c_{m-2}$, by similar reasoning as above we reach the contradiction $\xi_i < c_{m-1}$. Hence

$$c_{m-2} \leq t_{i-1} < t_i < c_{m+1},$$

and if $i \in N_1$

$$c_0 \leq t_{i-1} < t_i \leq c_2.$$

This allows the use of Henstock's Lemma in the next portion of the proof, namely for $m \geq 1$

$$\left| \sum_{i \in N_m} \left(HK \int_{t_{i-1}}^{t_i} f(t)\Delta t - f(\xi_i)(t_i - t_{i-1}) \right) \right| < \frac{\varepsilon}{2^{m+1}}.$$

Consider the following difference where the sum is taken over P,

$$\begin{aligned}
\left| I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| &< \frac{\varepsilon}{4} + \left| \int_a^A f(t)\Delta t - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| \\
&< \frac{\varepsilon}{2} + \left| \int_a^{t_n} f(t)\Delta t - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| \\
&= \frac{\varepsilon}{2} + \left| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(t)\Delta t - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right| \\
&= \frac{\varepsilon}{2} + \left| \sum_{i=1}^{n} \left(HK \int_{t_{i-1}}^{t_i} f(t)\Delta t - f(\xi_i)(t_i - t_{i-1}) \right) \right| \quad (3.6) \\
&= \frac{\varepsilon}{2} + \left| \sum_{i=1}^{\infty} \sum_{i \in N_m} \left(HK \int_{t_{i-1}}^{t_i} f(t)\Delta t - f(\xi_i)(t_i - t_{i-1}) \right) \right| \quad (3.7) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where the trivial result that any Δ-integrable function on a finite interval is $HK\Delta$-integrable has been used in (3.6) and Henstock's Lemma has been applied in (3.7), completing the proof.

□

Example 9: We show that if $\mathbb{T} = \mathbb{P}_{1,1} := \cup_{k=0}^{\infty}[2k, 2k+1]$, then

$$HK \int_0^{\infty} \frac{\Delta t}{(t+1)^2} = \ln(2) + \frac{\pi^2}{24}.$$

Proof. By Theorem 3.4 it suffices to show that the improper Δ-integral

$$\int_0^\infty \frac{\Delta t}{(t+1)^2} = \ln(2) + \frac{\pi^2}{24}.$$

To see this we consider

$$\begin{aligned}
\int_0^\infty \frac{\Delta t}{(t+1)^2} &= \sum_{k=0}^{\infty}\left(\int_{2k}^{2k+1}\frac{\Delta t}{(t+1)^2} + \int_{2k+1}^{2k+2}\frac{\Delta t}{(t+1)^2}\right)\\
&= \sum_{k=0}^{\infty}\left(\int_{2k}^{2k+1}\frac{dt}{(t+1)^2} + \int_{2k+1}^{2k+2}\frac{\Delta t}{(t+1)^2}\right)\\
&= \sum_{k=0}^{\infty}\left(-\frac{1}{t+1}\Big|_{2k}^{2k+1} + \frac{1}{(2k+2)^2}\right)\\
&= \sum_{k=0}^{\infty}\left(\frac{1}{(2k+1)(2k+2)} + \frac{1}{4(k+1)^2}\right)\\
&= \sum_{k=1}^{\infty}\left(\frac{1}{2k(2k-1)} + \frac{1}{4k^2}\right)\\
&= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{-\frac{1}{2}}{k(k-\frac{1}{2})} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\\
&= -\frac{1}{2}(-2\ln(2)) + \frac{1}{4}\frac{\pi^2}{6}\\
&= \ln(2) + \frac{\pi^2}{24}.
\end{aligned}$$

□

4 Convergence Theorems

In this section we offer theorems pertaining to taking limits through the integral sign of a $HK\Delta$-integral on an unbounded time scale interval.

Theorem 4.1 (Monotone Convergence Theorem). *Let* $f_n, f : [a,b]_{\mathbb{T}} \to \infty$ *and assume*

(i) f_n *is* $HK\Delta$*-integrable on* $[a,\infty)_{\mathbb{T}}$, $n \in \mathbb{N}$*;*

(ii) $f_n \to f$ *in* $[a,\infty)_{\mathbb{T}}$*;*

(iii) $f_n \le f_{n+1}$ *on* $[a,\infty)_{\mathbb{T}}$, $n \in \mathbb{N}$*;*

(iv) $I_n := HK\int_a^\infty f_n(t)\Delta t \to I$*.*

Then f is $HK\Delta$-integrable on $[a,\infty)_{\mathbb{T}}$ and

$$I = HK\int_a^\infty f(t)\Delta t.$$

Proof. This proof is formatted after a similar proof in [6]. Considering $f_n - f_1$ if necessary, assume $f_n(t)$ is nonnegative on $[a,\infty)_{\mathbb{T}}$. As in McLeod's proof, we wish to find a positive function g on $[a,\infty)_{\mathbb{T}}$ together with a Δ-gauge, δ^g, such that for all δ^g-fine partitions P of $[a,b]$ for any $b \in (a,\infty)_{\mathbb{T}}$,

$$\sum_{i=1}^{m} g(\xi_i)(t_i - t_{i-1}) \le 1.$$

Let G be Δ-differentiable on $\mathbb{T}$ such that $G^\Delta(t) > 0$ for all $t \in [a,\infty)_{\mathbb{T}}$ satisfying

$$0 < G(t) \le G(a) + \frac{1}{2}, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.1}$$

Set $g(t) = G^\Delta(t)$, for $t \in [a,\infty)_{\mathbb{T}}$.

We now define a Δ-gauge, $\delta^g = (\delta^g_L, \delta^g_R)$, on $[a,\infty)_{\mathbb{T}}$ as follows. If $\sigma(t) > t$, then $\delta^g_R(t) = \mu(t)$. If $\sigma(t) = t$, since G is differentiable at t, we can pick $\delta^g_R(t) > 0$, sufficiently small, so that

$$\frac{G(s) - G(t)}{s - t} \ge \frac{G^\Delta(t)}{2} = \frac{g(t)}{2}, \quad s \in [t, t + \delta^g_R(t)]_{\mathbb{T}}.$$

It follows from this if $\sigma(t) = t$, then

$$g(t)(s - t) \le 2[G(s) - G(t)], \quad s \in [t, t + \delta^g_R(t)]_{\mathbb{T}}. \tag{4.2}$$

It $t > a$ is left-dense, we can pick $\delta^g_L(t) > 0$, sufficiently small, so that

$$\frac{G(t) - G(s)}{t - s} \ge \frac{G^\Delta(t)}{2} = \frac{g(t)}{2}, \quad s \in [t - \delta^g_L(t), t]_{\mathbb{T}}.$$

It follows that if $\rho(t) = t$, then

$$g(t)(t - s) \le 2[G(t) - G(s)], \quad s \in [t - \delta^g_L(t), t]_{\mathbb{T}}. \tag{4.3}$$

Finally if $t > a$ and $\rho(t) < t$, then we define $\delta^g_L(t) = \frac{\nu(t)}{2}$.

Now let P be a δ^g-fine partition of $[a,b]_{\mathbb{T}}$ and consider the Riemann sum

$$\sum_{i=1}^{\infty} g(\xi_i)[t_i - t_{i-1}]. \tag{4.4}$$

We claim that each term in (4.4) satisfies

$$g(\xi_i)[t_i - t_{i-1}] \le 2[G(t_i) - G(t_{i-1})]. \tag{4.5}$$

There are many cases to consider here. First we consider the case where the tag point ξ_i is both right-dense and left-dense. In this case, using (4.2) and (4.3) we get

$$\begin{aligned} g(\xi_i)[t_i - t_{i-1}] &= g(\xi_i)[t_i - \xi_i] + g(\xi_i)[\xi_i - t_{i-1}] \\ &\leq 2[G(t_i) - G(\xi_i)] + 2[G(\xi_i) - G(t_{i-1})] \\ &= 2[G(t_i) - G(t_{i-1})]. \end{aligned}$$

Next consider the case when the tag point ξ_i is both left-scattered and right-scattered. In this case because of the way we defined δ^g we have that

$$t_{i-1} = \xi_i, \quad t_i = \sigma(\xi_i) = \sigma(t_{i-1}).$$

It follows that

$$g(\xi_i)[t_i - t_{i-1}] = G^{\Delta}(t_{i-1})[t_i - t_{i-1}] = G(t_i) - G(t_{i-1}) \leq 2[G(t_i) - G(t_{i-1})].$$

The remaining cases are left to the reader. Now using (4.5) we get the desired result

$$\sum_{i=1}^{m} g(\xi_i)[t_i - t_{i-1}] \leq 2\sum_{i=1}^{m}[G(t_i) - G(t_{i-1})] = 2[G(b) - G(a)] \leq 1$$

by (4.1).

We wish to prove that given $\varepsilon > 0$ there exists a Δ-gauge, δ, such that for all δ-fine partitions P, and all $n \in \mathbb{N}$

$$\left| I_n - \sum_{i=1}^{m} f_n(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon.$$

This implies that in the limit, on the same partition

$$\left| I - \sum_{i=1}^{m} f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon.$$

Let δ^n be a Δ-gauge such that for all δ^n-fine partitions P

$$\left| I_n - \sum_{i=1}^{m} f_n(\xi_i)(t_i - t_{i-1}) \right| < \frac{\varepsilon}{2^n}.$$

Fix a large integer N such that $|I_n - I_N| < \frac{\varepsilon}{4}$ for all $n \geq N$. Let $n \geq N$. Let F_n be the set of all $t \in [a, \infty)_{\mathbb{T}}$ such that $|f_k(t) - f(t)| < \frac{\varepsilon}{4} g(t)$ for all $k \geq n$ where $g(t)$ is as we constructed previously. By construction, we can see that $F_{n-1} \subset F_n$ and since f_n approaches f pointwise, $\cup_{i=N}^{\infty} F_n = [a, \infty)_{\mathbb{T}}$. Let $E_N = F_N$ and $E_n = F_n - F_{n-1}$ for all $n > N$. Note that each $t \in [a, \infty)_{\mathbb{T}}$ belongs to exactly one E_n. We now define the Δ-gauge, δ, such that

$$\delta_R(t) = min\{\delta_R^g(t), \delta_R^1(t), ..., \delta_R^n(t)\}$$

$$\delta_L(t) = min\{\delta_L^g(t), \delta_L^1(t), ..., \delta_L^n(t)\}$$

for $t \in E_n$.

$$\delta_L(\infty) = max\{\delta_L^g(\infty), \delta_L^1(\infty), ..., \delta_L^N(\infty)\}$$

at ∞. This defines δ on $[a, \infty]_{\mathbb{T}}$. We must now show that for all δ-fine partitions P,

$$\left| I_n - \sum_{i=1}^{m} f_n(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon$$

for all n. Let P be a δ-fine partition of $[a, \infty]_{\mathbb{T}}$. Since P is δ-fine, it is also δ^n-fine for all $n \le N$. Therefore

$$\left| I_n - \sum_{i=1}^{m} f_n(\xi_i)(t_i - t_{i-1}) \right| < \frac{\varepsilon}{2^n}$$

for $n \le N$. The more difficult case is $n > N$. Let $\nu_n([u,v]) = \int_u^v f_n(t)\Delta t$. Define subsets of P as follows: Let $P_i = \{([u,v], \xi) \in P : \xi \in E_i\}$. Let $\mathcal{E} = \cup_{i=N}^{n-1} P_i$ and $\mathcal{F} = \cup_{i=n}^{\infty} P_i$. Note that since P_i is δ^j-fine for all $j \le i$, $\mathcal{F}$ is δ^n-fine but $\mathcal{E}$ is not. Now consider

$$\begin{aligned} \left| I_n - \sum_{i=1}^{m} f_n(\xi_i)(t_i - t_{i-1}) \right| &\le \left| \sum_{\mathcal{E}} \Big(\nu_n([u,v]) - f_n(\xi)(v-u) \Big) \right| + \left| \sum_{\mathcal{F}} \Big(\nu_n([u,v]) - f_n(\xi)(v-u) \Big) \right| \\ &\le \left| \sum_{\mathcal{E}} \Big(\nu_n([u,v]) - f_n(\xi)(v-u) \Big) \right| + \frac{\varepsilon}{2^n}. \end{aligned}$$

This follows from Henstock's Lemma.

$$\begin{aligned} \left| \sum_{\mathcal{E}} \Big(\nu_n([u,v]) - f_n(\xi)(v-u) \Big) \right| &\le \sum_{i=N}^{n-1} \left| \sum_{P_i} \Big(\nu_n([u,v]) - \nu_i([u,v]) \Big) \right| \\ &+ \sum_{i=N}^{n-1} \left| \sum_{P_i} \Big(\nu_i([u,v]) - f_i(\xi)(v-u) \Big) \right| \\ &+ \sum_{i=N}^{n-1} \left| \sum_{P} \Big(f_i(\xi)(v-u) - f_n(\xi)(v-u) \Big) \right|. \end{aligned}$$

For the first sum,

$$\begin{aligned}\sum_{i=N}^{n-1}\left|\sum_{P_i}\Big(\nu_n([u,v])-\nu_i([u,v])\Big)\right| &\le \sum_{\mathcal{E}}\Big|\nu_n([u,v])-\nu_N([u,v])\Big| \\ &\le |I_n-I_N| \\ &< \frac{\varepsilon}{4}.\end{aligned}$$

The first two inequalities are results of the monotonicity and nonnegativity of the sequence $\{f_n\}$, while the third results from our choice of N. For the second sum,

$$\sum_{i=N}^{n-1}\left|\sum_{P_i}\Big(\nu_i([u,v])-f_i(\xi)(v-u)\Big)\right|\le\sum_{i=N}^{n-1}\frac{\varepsilon}{2^n}<\frac{\varepsilon}{4}.$$

This again follows from Henstock's Lemma. For the third sum,

$$\sum_{i=N}^{n-1}\left|\sum_{P_i}\Big(f_i(\xi)(v-u)-f_n(\xi)(v-u)\Big)\right|\le\sum_{\mathcal{E}}\frac{\varepsilon}{4}g(\xi)(v-u)\le\frac{\varepsilon}{4}.$$

The first inequality results from our definition of E_i while the second results from our δ-fine partition also being δ^g-fine as well as our construction of g. Therefore

$$\left|\sum_{\mathcal{E}}\Big(\nu_n([u,v])-f_n(\xi)(v-u)\Big)\right|<\frac{3\varepsilon}{4}.$$

Hence

$$\left|I_n-\sum_{i=1}^{m}f(\xi_i)(t_i-t_{i-1})\right|<\frac{3\varepsilon}{4}+\frac{\varepsilon}{2^n}<\varepsilon.$$

Therefore we may conclude that $|I-\sum_{i=1}^{m}f(\xi_i)(t_i-t_{i-1})|<\varepsilon$ for all δ-fine partitions of $[a,\infty)_{\mathbb{T}}$. □

Theorem 4.2 (Dominated Convergence Theorem). *Suppose f_n, $n\in\mathbb{N}$, g, and h are $HK\Delta$-integrable on $[a,\infty)_{\mathbb{T}}$ and $g(t)\le f_n(t)\le h(t)$ for $t\in[a,\infty)_{\mathbb{T}}$. Also supposed $f_n\to f$ pointwise on $[a,\infty)_{\mathbb{T}}$. Then f is $HK\Delta$-integrable on $[a,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty}HK\int_a^\infty f_n(t)\Delta t=HK\int_a^\infty f(t)\Delta t$.*

For the proof of this theorem, we shall use the following lem.

Lemma 4.3. *If f_k, $1\le k\le n$, and h are $HK\Delta$-integrable on $[a,\infty)_{\mathbb{T}}$, with $0\le f_k(t)\le h(t)$ for $t\in[a,\infty)_{\mathbb{T}}$, and $1\le k\le n$, then $f(t):=max\{f_1(t),f_2(t),...,f_n(t)\}$ is $HK\Delta$-integrable on $[a,\infty)_{\mathbb{T}}$.*

Proof. If this is true for $n = 2$, then we may proceed by induction to show that it is true for arbitrary n. Since $f_k(t) \le h(t)$, the maximum of any pair of functions is also $\le h(t)$. Therefore it suffices to prove the lem for $n = 2$. Suppose f, g, and h are $HK\Delta$-integrable on $[a, \infty)_{\mathbb{T}}$ and $f(t), g(t) \le h(t)$. Consider

$$|f(t) - g(t)| = 2\, max\{f(t), g(t)\} - f(t) - g(t) \le 2h(t) - f(t) - g(t)$$

Since $2h(t) - f(t) - g(t)$ is $HK\Delta$-integrable, $|f(t) - g(t)|$ is $HK\Delta$-integrable. Therefore $max\{f(t), g(t)\}$ is also $HK\Delta$-integrable. □

Proof. (of the Dominated Convergence Theorem) Note here as in the Monotone Convergence Theorem, that we may consider only nonnegative functions bounded by an integrable upper limit by considering $h(t)-g(t)$ and $f_n(t)-g(t)$. In the proof of the Monotone Convergence Theorem, we only use the monotone convergence of the sequence of functions to determine how large N must be for $|I_n - \sum_{i=1}^{m} f_n(\xi_i)(t_i - t_{i-1})| < \varepsilon$ for all n. Therefore, given our suppositions on the convergent sequence of functions, we shall simply give a new definition of N such that the term,

$$\sum_{i=N}^{n-1} \left| \sum_{P_i} \Big(\nu_n([u, v]) - \nu_i([u, v]) \Big) \right| < \frac{\varepsilon}{4},$$

where P_i is as it is in the proof of the Monotone Convergence Theorem. Since $0 \le f_k(t) \le h(t)$ for all k, $|f_m(t) - f_n(t)| \le h(t)$ for all m, n. By the previous lem, $|f_m(t) - f_n(t)|$ is $HK\Delta$-integrable and so for all j, k, $g_{j,k}(t) = max_{j \le m,n \le k}\{|f_m(t) - f_n(t)|\}$ is $HK\Delta$-integrable and $g_{j,k}(t) \le h(t)$. We apply the Monotone Convergence Theorem to $\{g_{j,k}\}_{k=1}^{\infty}$ to say that $g_{j,k} \to p_j$ where p_j is $HK\Delta$-integrable. By the Cauchy criterion, $p_j(t) \to 0$ for all t. We may apply the Monotone Convergence Theorem again to $-p_j(t)$ to say that $\int_a^{\infty} p_j(t)\Delta t \to 0$ as $j \to \infty$. Therefore we choose N such that $\int_a^{\infty} p_n(t)\Delta t < \frac{\varepsilon}{4}$ for all $n \ge N$. We have

$$\sum_{i=N}^{n-1} \left| \sum_{P_i} \Big(\nu_n([u, v]) - \nu_i([u, v]) \Big) \right| \le \sum_{\mathcal{E}} \int_u^v p_n(t)\Delta t \le \int_a^{\infty} p_n(t)\Delta t < \frac{\varepsilon}{4},$$

where $\mathcal{E} = \cup_{i=N}^{n-1} P_i$. This concludes the proof. □

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