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Exponential Stability of Dynamic Equations on Time Scales

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Abstract

We will be concerned with stability theory for dynamic equations on time scales. Our main result deals with the stability of a perturbed linear dynamic equation. We give several examples where our theorem applies, including an application to the dynamic logistic equation.

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1 Introduction

The study of dynamic equations on time scales was created by Stefan Hilger [4] to unify the calculus of differential and difference equations, and to extend these to the calculus on time scales. A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The three most common examples of calculus on time scales are differential calculus, difference calculus [6], and quantum calculus [5], i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : \; t \in \mathbb{N}_0\}$, where $q > 1$. Dynamic equations on time scales have great potential for applications such as in population dynamics. For example, take an insect population that dies every October 1st leaving its eggs behind and the eggs hatch every April 1st. A model of this population would include a continuous portion from April 1st to October 1st and a discrete portion the rest of the year. A model without time scales would have to split this into two cases and analyze each. With time scales one model could handle both portions of the year. A time scale for this could be $\mathbb{T} = \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1]$. A cover story article in New Scientist [7] highlighted the new possibilities that exist for modeling natural systems using time scales.

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Some of these applications include models of the West Nile virus, the stock market, and any population models that vary in continuous time and discrete time.

The following general results come from the work of Hilger [4] and appear in the books by Bohner and Peterson [1] and [2]. For $t \in \mathbb{T}$, we define the **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$$

while the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set [1]. A point $t \in \mathbb{T}$ is defined to be **left–dense** if $\rho(t) = t$ and $t > \inf \mathbb{T}$, and is **right–dense** if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, and is **left–scattered** if $\rho(t) < t$ and is **right–scattered** if $\sigma(t) > t$. Points that are right-scattered and left-scattered at the same time are called **isolated** (also a left-scattered maximum and a right-scattered minimum are said to be isolated). A function $g : \mathbb{T} \to \mathbb{R}$ is said to be **right–dense continuous** (rd–continuous) if g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The **graininess function** μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. Define $\mathbb{T}^\kappa$ to be $\mathbb{T} - M$ if $\mathbb{T}$ has a left-scattered maximum M, otherwise let $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1.1. Fix $t \in \mathbb{T}^\kappa$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^\Delta(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^\Delta(t)$ is the **(delta) derivative** of x at t and that x is **(delta) differentiable** at t.

Theorem 1.1. *Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*
(i) If g is differentiable at t, then g is continuous at t.
(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}.$$

(iii) If g is differentiable and t is right-dense, then

$$g^\Delta(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^\Delta(t)$.

In this paper we will refer to the (delta) integral which we can define as follows:

Definition 1.2. If $G^\Delta(t) = g(t)$, then the **Cauchy (delta) integral** of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It can be shown that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^{t} g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. For a more general definition of the delta integral see [1] and [2].

Definition 1.3. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is ***regressive*** provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Definition 1.4. We define the set $\mathcal{R}^+$ of all ***positively regressive*** elements of $\mathcal{R}$ by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Theorem 1.2. *If we define* **circle plus** *addition* $\oplus$ *on* $\mathcal{R}$ *by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad t \in \mathbb{T}$$

then $(\mathcal{R}, \oplus)$ *is an Abelian group. The additive inverse of p under the operation* $\oplus$ *is defined by*

$$\ominus p(t) := -\frac{p(t)}{1 + \mu(t)p(t)}.$$

Definition 1.5. We define ***circle minus*** subtraction $\ominus$ on $\mathcal{R}$ by

$$p \ominus q := p \oplus (\ominus q).$$

Definition 1.6 (Hilger [4]). For $h > 0$, define $\mathbb{Z}_h$ by

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} | -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\},$$

and define $\mathbb{C}_h$ by

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} | z \neq -\frac{1}{h} \right\}.$$

For $h = 0$, let $\mathbb{Z}_0 = \mathbb{C}_0 = \mathbb{C}$, the set of complex numbers.

Definition 1.7 (Cylinder Transformation). For $h \geq 0$, we define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \begin{cases} \frac{1}{h} Log(1 + zh), & \text{if } h > 0 \\ z, & \text{if } h = 0, \end{cases}$$

where *Log* is the principal logarithm function.

Definition 1.8 (Exponential Function). If $p \in \mathcal{R}$, then we define the exponential function by

$$e_p(t,s) = exp\left(\int_s^t \xi_{\mu(\tau)}\left(p\left(\tau\right)\right)\Delta\tau\right) \quad \text{for} \quad s,t \in \mathbb{T},$$

where the cylinder transformation $\xi_h(z)$ is defined in Definition 1.7.

Theorem 1.3 (Properties of the Exponential Function). *If $p,\ q \in \mathcal{R}$ and $t,\ r,\ s \in \mathbb{T}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))\, e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

Next are the two Variations of Constants formulas [1], each corresponding to one of the two forms of a first order linear dynamic equation.

Theorem 1.4 (Variation of Constants). *Suppose f is rd−continuous on $\mathbb{T}$ and $p \in \mathcal{R}$, then the unique solution of the initial value problem*

$$x^{\Delta} = -p(t)\, x^{\sigma} + f(t), \quad x(t_0) = x_0$$

where $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$, is given by

$$x(t) = e_{\ominus p}(t,t_0)\, x_0 + \int_{t_0}^t e_{\ominus p}(\,t,\tau)f(\tau)\Delta\tau.$$

Theorem 1.5 (Variation of Constants). *Suppose f is rd−continuous on $\mathbb{T}$ and $p \in \mathcal{R}$. If $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$, then the unique solution of the initial value problem*

$$x^{\Delta} = p(t)x + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_p(t,t_0)x_0 + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

Theorem 1.6 (Gronwall's Inequality). *Let $y, f \in C_{rd}$ and $g \in \mathcal{R}^+$, $g \geq 0$. Then*

$$y(t) \leq f(t) + \int_{t_0}^t y(\tau)\, g(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^t e_g(t,\sigma(\tau))\, f(\tau)g(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}.$$

Theorem 1.7 (Bernoulli's Inequality). *Let $\alpha \in \mathbb{R}$ with $\alpha \in \mathcal{R}^+$. Then*

$$e_\alpha(t,s) \geq 1 + \alpha(t-s) \text{ for all } t \geq s.$$

2 Main Results

For $p \in \mathcal{R}$ we define

$$\beta_p(t) = \begin{cases} \frac{1}{\mu(t)} \log|1+\mu(t)p(t)|, & \mu(t) > 0, \\ p(t), & \mu(t) = 0, \end{cases}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

The following result is a slight generalization of a result due to Gard and Hoffacker [3] which we state and prove here for convenience.

Theorem 2.1. *Let $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, and assume that $r \leq \beta_p(t) \leq q$, for $t \in [t_0, \infty)_{\mathbb{T}}$. Then*

$$e^{r(t-t_0)} \leq |e_p(t,t_0)| \leq e^{q(t-t_0)}$$

for all $t \in [\,t_0, \infty)_{\mathbb{T}}$.

Proof. This result follows easily from the formula

$$\begin{aligned} |e_p(t,t_0)| &= |e^{\int_{t_0}^t \xi_{\mu(s)}(p(s))\Delta s}| \\ &= e^{\int_{t_0}^t \Re\left(\xi_{\mu(s)}(p(s))\right)}\Delta s \\ &= e^{\int_{t_0}^t \beta_p(s)\Delta s}, \end{aligned}$$

and the fact that $r \leq \beta_p(t) \leq q$, for $t \in [t_0, \infty)_{\mathbb{T}}$. □

Now consider the dynamic equation

$$x^{\Delta} = g\,(t,x)\,, \tag{2.1}$$

where we assume that solutions of initial value problems for (2.1) are unique and exist on the whole time scale interval $[t_0, \infty)_{\mathbb{T}}$, which we assume is unbounded above. We let $x\,(t, t_1, x_1)$ denote the unique solution of the initial value problem (2.1), $x(t_1) = x_1$. For convenience, we assume $g\,(t, 0) = 0$ so that $x(t) \equiv 0$ is a solution (called the trivial solution) of (2.1). Next we define the different types of stability that will be of interest to us in this paper.

Definition 2.1. The trivial solution of (2.1) is ***stable*** on $[\,t_0, \infty)_{\mathbb{T}}$ provided given any $t_1 \in [\,t_0, \infty)_{\mathbb{T}}$ and $\epsilon > 0$, there is a $\delta = \delta\,(t_1, \epsilon) > 0$ such that if $|x_1| < \delta$, then $|x\,(t, t_1, x_1)| < \epsilon$ for all $t \in [\,t_1, \infty)_{\mathbb{T}}$. The trivial solution of (2.1) is ***asymptotically stable*** on $[\,t_0, \infty)_{\mathbb{T}}$ provided it is stable on $[\,t_0, \infty)_{\mathbb{T}}$ and given any $t_1 \in [t_0, \infty)_{\mathbb{T}}$ there is a $\delta_1 = \delta_1(t_1) > 0$ such that if $|x_1| < \delta_1$, then

$$\lim_{t \to \infty} x(t, t_1, x_1) = 0.$$

In the latter case the trivial solution of (2.1) is ***exponentially asymptotically stable*** on $[t_0,\infty)_{\mathbb{T}}$ provided given any $t_1 \in [t_0,\infty)_{\mathbb{T}}$, there is a $K = K(t_1) > 0$, $\delta_1 > 0$, and $q > 0$ such that

$$|x(t,t_1,x_1)| \leq Ke^{-q(t-t_1)}|x_1|$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. If K does not depend on t_1, then we say that the trivial solution is ***uniformly exponentially asymptotically stable*** on $[t_0,\infty)_{\mathbb{T}}$.

We will be concerned with the almost linear dynamic equation

$$x^\Delta = p(t)x + f(t,x), \tag{2.2}$$

where we assume throughout that solutions of initial value problems for (2.2) are unique and exist on the whole time scale interval $[t_0,\infty)_{\mathbb{T}}$, and we will let $x(t) = x(t,t_1,x_1)$ denote the unique solution of the IVP (2.2), $x(t_1) = x_1$.

The following theorem corresponds to Theorem 5.3 in [3] where they seem to ignore a certain expression (see (2.4)) which could be unbounded.

Theorem 2.2. *Let $[t_0,\infty)_{\mathbb{T}}$ be unbounded above, and let N be a neighborhood of $x = 0$. Assume $f(t,x)$ is a real-valued continuous function for $(t,x) \in [t_0,\infty)_{\mathbb{T}} \times N$ which satisfies the condition*

$$\lim_{x\to 0} \frac{f(t,x)}{x} = 0 \tag{2.3}$$

uniformly for $t \in [t_0,\infty)_{\mathbb{T}}$. Also assume

$$\frac{1}{|1+\mu(t)p(t)|} \leq M \tag{2.4}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$ for some $M \in \mathbb{R}^+$. If $p \in \mathcal{R}$ and $q := \limsup_{t\to\infty} \beta_p(t) < 0$, then the trivial solution of (2.2) is exponentially asymptotically stable on $[t_0,\infty)_{\mathbb{T}}$. Furthermore, if $\bar{q} := \sup\{\beta_p(t) : t \in [t_0,\infty)_{\mathbb{T}}\} < 0$, then the trivial solution of equation (2.2) is uniformly exponentially asymptotically stable on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let $x(t) = x(t,t_1,x_1)$, then by the variation of constants formula in Theorem 1.5 we have

$$\begin{aligned}
x(t) &= e_p(t,t_1)x_1 + \int_{t_1}^t e_p(t,\sigma(\tau))f(\tau,x(\tau))\Delta\tau \\
e_{\ominus p}(t,t_1)x(t) &= x_1 + \int_{t_1}^t e_{\ominus p}(t,t_1)e_p(t,\sigma(\tau))f(\tau,x(\tau))\Delta\tau \\
&= x_1 + \int_{t_1}^t e_{\ominus p}(t,t_1)e_{\ominus p}(\sigma(\tau),t)f(\tau,x(\tau))\Delta\tau \\
&= x_1 + \int_{t_1}^t e_{\ominus p}(\sigma(\tau),t_1)f(\tau,x(\tau))\Delta\tau.
\end{aligned}$$

From (2.3), we have given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(t,x)| \leq \epsilon|x|$ for $t \in [t_0,\infty)_{\mathbb{T}}$, $|x| < \delta$. Now assume $|x_1| < \delta$, then for those $t \in [t_1,\infty)$ such that $|x(t)| < \delta$

we have

$$\begin{aligned}
|e_{\ominus p}(t,t_1)x(t)| &\leq |x_1| + \int_{t_1}^{t} |e_{\ominus p}(\sigma(\tau),t_1)||f(\tau,x(\tau))|\Delta\tau \\
&\leq |x_1| + \epsilon\int_{t_1}^{t} |e_{\ominus p}(\sigma(\tau),t_1)||x(\tau)|\Delta\tau \\
&= |x_1| + \epsilon\int_{t_1}^{t} |e_{\ominus p}(\sigma(\tau),t_1)x(\tau)|\Delta\tau \\
&= |x_1| + \epsilon\int_{t_1}^{t} |[1+(\ominus p)(\tau)\mu(\tau)]e_{\ominus p}(\tau,t_1)x(\tau)|\Delta\tau \\
&= |x_1| + \epsilon\int_{t_1}^{t} \frac{1}{|1+\mu(\tau)p(\tau)|}|e_{\ominus p}(\tau,t_1)x(\tau)|\Delta\tau.
\end{aligned}$$

In Gronwall's Inequality (Theorem 1.6) we let $f(t) = |x_1|$, $g(t) = \frac{\epsilon}{|1+\mu(t)p(t)|}$, and $y(t) = |e_{\ominus p}(t,t_1)x(t)|$. We get

$$\begin{aligned}
|e_{\ominus p}(t,t_1)x(t)| &\leq |x_1| + \int_{t_1}^{t} e_g(t,\sigma(\tau))|x_1|g(\tau)\Delta\tau \\
&= |x_1| + |x_1|\int_{t_1}^{t} e_{\ominus g}(\sigma(\tau),t)g(\tau)\Delta\tau \\
&= |x_1| + |x_1|\int_{t_1}^{t} [1+\mu(\tau)(\ominus g)(\tau)]e_{\ominus g}(\tau,t_1)g(\tau)\Delta\tau \\
&= |x_1| + |x_1|\int_{t_1}^{t} \frac{1}{1+g(\tau)\mu(\tau)}e_{\ominus g}(\tau,t)g(\tau)\Delta\tau \\
&= |x_1| - |x_1|\int_{t_1}^{t} (\ominus g)(\tau)e_{\ominus g}(\tau,t)\Delta\tau \\
&= |x_1| - |x_1|\left[e_{\ominus g}(\tau,t)\right]_{\tau=t_1}^{\tau=t} \\
&= |x_1| - |x_1|[e_{\ominus g}(t,t) - e_{\ominus g}(t_1,t)] \\
&= |x_1|e_g(t,t_1).
\end{aligned}$$

Therefore,

$$|x(t)| \leq |x_1||e_p(t,t_1)|e_g(t,t_1)$$

as long as $|x(t)| < \delta$.

Since $\frac{1}{|1+\mu(\tau)p(\tau)|} \leq M$ for $t \in [t_0,\infty)_{\mathbb{T}}$ for some $M \in \mathbb{R}^+$ we get

$$\begin{aligned}
e_g(t,t_1) &= e_{\frac{\epsilon}{|1+\mu(t)p(\tau)|}}(t,t_1) = e^{\int_{t_1}^{t}\xi_{\mu(\tau)}\left(\frac{\epsilon}{|1+\mu(\tau)p(\tau)|}\right)\Delta\tau} \\
&\leq e^{\int_{t_1}^{t}\left(\frac{\epsilon}{|1+\mu(\tau)p(\tau)|}\right)\Delta\tau} \quad \text{(by Definition 1.7)} \\
&\leq e^{\int_{t_1}^{t} M\epsilon\Delta\tau} \\
&= e^{M\epsilon(t-t_1)}.
\end{aligned}$$

Therefore,

$$|x(t)| \leq |e_p(t, t_1)| e^{M\epsilon(t-t_1)} |x_1|.$$

Let $q := \limsup_{t\to\infty} \beta_p(t) < q_1 < 0$, then there exists a $T_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $\beta_p(t) \leq q_1 < 0$ for all $t \in [T_1, \infty)_{\mathbb{T}}$.

Using this, Theorem 1.3, and Theorem 2.1 we have for $t \in [t_1, \infty)_{\mathbb{T}}$

$$\begin{aligned} |x(t)| &\leq |e_p(t, t_1)| e^{M\epsilon(t-t_1)} |x_1| \\ &= |e_p(t, T_1)| |e_p(T_1, t_1)| e^{M\epsilon(t-t_1)} ||x_1| \\ &\leq e^{q_1(t-T_1)} |e_p(T_1, t_1)| e^{M\epsilon(t-t_1)} |x_1|. \end{aligned}$$

Now let

$$K = \frac{|e_p(T_1, t_1)|}{e^{q_1(T_1-t_1)}},$$

then

$$\begin{aligned} |x(t)| &\leq K e^{q_1(t-t_1)} e^{M\epsilon(t-t_1)} |x_1| \\ &\leq K e^{(q_1+M\epsilon)(t-t_1)} |x_1| \end{aligned} \tag{2.5}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$ as long as $|x(t)| < \delta$. Since $q_1 < 0$, choose $\epsilon > 0$, sufficiently small, so that $q_1 + M\epsilon < 0$. Note here that if $|x_1|$ is sufficiently small then $|x(t)| < \delta$ for $t \in [t_1, \infty)_{\mathbb{T}}$, so the above steps are correct. But then we get that equation (2.2) is exponentially asymptotically stable on $[t_0, \infty)_{\mathbb{T}}$.

Now suppose $\bar{q} := \sup\{\beta_p(t) : t \in [t_0, \infty)_{\mathbb{T}}\} < 0$. This implies $\beta_p(t) \leq \bar{q} < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Let $t_1 \in [t_0, \infty)_{\mathbb{T}}$, then by Theorem 2.1 we have

$$|e_p(t, t_1)| \leq e^{\bar{q}(t-t_1)}.$$

From this and (2.5),

$$\begin{aligned} |x(t)| = |x(t, t_1, x_1)| &\leq K e^{\bar{q}(t-t_1)} e^{M\epsilon(t-t_1)} |x_1| \\ &= K e^{(\bar{q}+M\epsilon)(t-t_1)} |x_1|. \end{aligned}$$

Since $\bar{q} < 0$, we can choose $\epsilon > 0$, sufficiently small, so that $q + M\epsilon < 0$. Therefore, the trivial solution of (2.2) is uniformly exponentially asymptotically stable on $[t_0, \infty)_{\mathbb{T}}$. □

Once we see the proof of Theorem 2.3 it is easy to prove the following result concerning the linear homogeneous dynamic equation

$$x^{\Delta} = p(t)x. \tag{2.6}$$

Theorem 2.3. *Assume $p \in \mathbb{T}$. If $\limsup_{t\to\infty} \beta_p(t) = q < 0$, then the trivial solution of (2.6) is exponentially asymptotically stable on $[t_0, \infty)_{\mathbb{T}}$. If $\beta_p(t) \leq q < 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$, then the trivial solution of (2.6) is unifomly exponentially asymptotically stable on $[t_0, \infty)_{\mathbb{T}}$. If $\liminf_{t\to\infty} \beta_p(t) = q > 0$, then the trivial solution of (2.6) is unstable on $[t_0, \infty)_{\mathbb{T}}$.*

As a corollary we now get a result due to Gard and Hoffacker [3].

Corollary 2.4. *Suppose p is a regressive constant with $p < 0$. If*

$$\limsup_{t\to\infty} \mu(t) < -\frac{2}{p},$$

then the trivial solution of (2.6) *is exponentially asymptotically stable on* $[t_0, \infty)_{\mathbb{T}}$. *If*

$$\sup\{\mu(t) : t \in [t_0, \infty)_{\mathbb{T}}\} < -\frac{2}{p},$$

then the trivial solution of (2.6) *is uniformly exponentially asymptotically stable on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. For a constant $p \in \mathcal{R}$ define the function α by

$$\alpha(\mu) = \begin{cases} \frac{1}{\mu}\log|1+\mu p|, & \mu > 0, \quad \mu \neq -\frac{1}{p}, \\ p, & \mu = 0. \end{cases}$$

Note that when p is a constant

$$\beta_p(t) = \alpha(\mu(t)), \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

For $p < 0$, the graph of the function α is given in Figure 1.

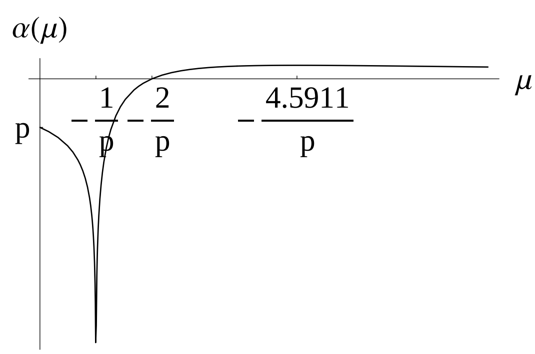

Figure 1: Graph of $y = \alpha(\mu)$ where the constant $p < 0$.

From this graph we see that

$$\limsup_{t\to\infty} \mu(t) < -\frac{2}{p},$$

implies that

$$\limsup_{t\to\infty} \beta_p(t) = \limsup_{t\to\infty} \alpha(\mu(t)) < 0.$$

Also

$$\sup\{\mu(t), t \in [t_0, \infty)_{\mathbb{T}}\} < \frac{-2}{p}$$

implies that

$$\sup\{\beta_p(t), t \in [t_0, \infty)_{\mathbb{T}}\} < 0.$$

The conclusions of this corollary then follow from Theorem 2.3. □

Similar to the proof of Corollary 2.4 we can use Theorem 2.1 to prove the following result. This result appears in [3] but we need to assume (2.4) holds and we do not see how to eliminate this assumption.

Corollary 2.5. *Suppose* $p(t) \equiv p$ *is a regressive constant with* $p < 0$*. Assume* (2.3) *and* (2.4) *hold. If* $\limsup_{t\to\infty} \mu(t) < -\frac{2}{p}$*, then the trivial solution of* (2.2) *is exponentially asymptotically stable on* $[t_0, \infty)_{\mathbb{T}}$*. If* $\sup\{\mu(t) : t \in [t_0, \infty)_{\mathbb{T}}\} < -\frac{2}{p}$*, then the trivial solution of* (2.2) *is uniformly exponentially asymptotically stable on* $[t_0, \infty)_{\mathbb{T}}$*.*

We now give a result where the linearized equation is unstable implies the almost linear system (2.2) is unstable.

Theorem 2.6. *Let* $[t_0, \infty)_{\mathbb{T}}$ *be unbounded and let* N *be a neighborhood of* $x = 0$*. Assume* $f(t, x)$ *is a real-valued, continuous function for* $(t, x) \in \mathbb{T} \times N$ *and*

$$\lim_{x\to 0} \frac{f(t, x)}{x} = 0$$

uniformly for t *in* $\mathbb{T}$*. Suppose* $p \in \mathcal{R}$ *and* $\liminf_{t\to\infty} p(t) > 0$*. Then the trivial solution of* (2.2) *is unstable on* $[t_0, \infty)_{\mathbb{T}}$*.*

Proof. Since $\liminf_{t\to\infty} p(t) > 0$ and $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$ uniformly on $[t_0, \infty)_{\mathbb{T}}$, there is a $\bar{p} > 0$, a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $p(t) \geq \bar{p}$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and if $0 < \epsilon < \bar{p}$, then there is a $\delta_1 > 0$ such that if $|x| < \delta_1$ then $|f(t, x)| \leq \epsilon|x|$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Suppose the trivial solution of (2.2) is stable on $[t_0, \infty)_{\mathbb{T}}$. Then there exists a δ_2 satisfying $0 < \delta_2 \leq \delta_1$ such that if $|x_1| < \delta_2$, then $|x(t) := x(t, t_1, x_1)| < \delta_1$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Fix $0 < x_1 < \delta_2$, then $x(t_1) = x_1 > 0$. We claim that $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. If we assume not then there is a point $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$x(t_2) \leq 0, \quad \text{and} \quad x(t) > 0, \quad t \in [t_1, t_2)_{\mathbb{T}}.$$

Then

$$\begin{aligned} x^{\Delta}(t) &= p(t)x(t) + f(t, x(t)) \\ &\geq p(t)x(t) - \epsilon\,|x(t)| \\ &\geq (p(t) - \epsilon)x(t) \\ &\geq (\bar{p} - \epsilon)x(t) \end{aligned} \tag{2.7}$$

for $t \in [t_1, t_2)_{\mathbb{T}}$. But this implies $x(t_2) > 0$ which is a contradiction. Hence $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ and hence (2.7) holds on $[t_1, \infty)_{\mathbb{T}}$. But then it follows that

$$x(t) = x(t, t_1, x_1) \geq e_{\bar{p}-\epsilon}(t, t_1)x_1, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

which contradicts the fact that $|x(t)| \leq \delta_1$ for $t \in [t_1, \infty)_{\mathbb{T}}$.

□

3 Examples

In population dynamics, the logistic equation is used frequently to model changes in population. It is written

$$x^{\Delta} = [a \ominus (gx)]\, x, \tag{3.1}$$

where $gx \in \mathcal{R}$. Note that (3.1) can be written in the form

$$x^{\Delta} = \frac{a(t)}{1+\mu(t)g(t)x}x - \frac{g(t)}{1+\mu(t)g(t)x}x^2.$$

It is pointed out in [2] why this is the true logistic equation for time scales.

Example 3.1 *(Logistic Equation).* Suppose $a \in \mathcal{R}$ is a constant and $N > 0$ is a constant, and define

$$g = \frac{1}{N}a, \quad N > 0.$$

In this case the dynamic logistic equation is

$$x^{\Delta} = \frac{a}{1+\mu(t)\frac{a}{N}x}x - \frac{a}{N(1+\mu(t)\frac{a}{N}x)}x^2. \tag{3.2}$$

If $x(t)$ is the population of some species at time t, then the constant a is called the ***intrinsic growth rate*** and K is called the ***saturation level*** or ***enviromental carrying capacity*** of the population.

We are interested in the equilibrium solution

$$x(t) = N.$$

Since our results concern the stability of the trivial solution we let

$$y(t) := \frac{a}{g} - x(t) = N - x(t),$$

where x is a solution of (3.2). It can be shown that y then solves the dynamic equation

$$y^{\Delta} = (\ominus a)(t)y + \frac{ay^2}{N(1+\mu(t)a)(1+\mu(t)a - \frac{a\mu(t)}{N}y)}, \tag{3.3}$$

which is of the form (2.2).

Consider the time scale $\mathbb{T} = h\mathbb{Z}$. Let $p = (\ominus a)(t) = \frac{-a}{1+\mu(t)a} = -\frac{a}{1+ha}$. Then it can be shown that $p < 0$ and $\mu(t) = h < -\frac{2}{p}$ iff $a < -\frac{2}{h}$ or $a > 0$; and $p > 0$ iff $-\frac{1}{h} < a < 0$. Hence by Corollary 2.5, we have that if $a < -\frac{2}{h}$ or $a > 0$, (3.3) is uniformly exponentially stable; and by Theorem 2.6 if $-\frac{1}{h} < a < 0$, (3.3) is unstable.

Note that neither Theorem 2.2 nor Corollary 2.5 applies to this example when

$$-\frac{2}{h} < a < -\frac{1}{h}.$$

But, in this case, we can use Corollary 2.27 in [2] we get that if $x_0 > 0$, then the solution of the IVP (3.2), $x(t_0) = x_0$ is given by

$$\begin{aligned} x(t) &= \frac{1}{\frac{1}{N} + \left(\frac{1}{x_0} - \frac{1}{N}\right) e_{\ominus p}(t, t_0)} \\ &= \frac{1}{\frac{1}{N} + \left(\frac{1}{x_0} - \frac{1}{N}\right) e_a(t, t_0)} \\ &= \frac{1}{\frac{1}{N} + \left(\frac{1}{x_0} - \frac{1}{N}\right) (1 + ah)^{\frac{t-t_0}{h}}}. \end{aligned}$$

Note that since $-1 < 1 + ah < 0$, $\lim_{t\to\infty} x(t) = N$. Hence we see that the equilibrium solution $x(t) = N$ is asymptotically stable if

$$-\frac{2}{h} < a < -\frac{1}{h}.$$

Example 3.2. Consider the dynamic equation

$$x^{\Delta} = px - x^2 \tag{3.4}$$

with the time scale composed of infinitely many copies of the Cantor set C, that is

$$\mathbb{T} = \bigcup_{n=0}^{\infty} \{t = n + c, \quad c \in C\}$$

and $p = -\frac{1}{3}$. Note that $\sup \mu(t) = \frac{1}{3}$ and that $\inf \mu(t) = 0$. Observe that all the conditions of Corollary 2.5 are satisfied:

$$\begin{aligned} p &< 0, \\ \sup \mu(t) &< -\frac{2}{p} = 6, \end{aligned}$$

and

$$\frac{1}{|1 + \mu(t)p|} = \frac{1}{\left|1 - \frac{1}{3}\mu(t)\right|},$$

where this quantity reaches its maximum at $\sup \mu(t) = \frac{1}{3}$. Thus

$$\frac{1}{\left|1 - \frac{1}{3}\mu(t)\right|} \leq \frac{9}{8} = M$$

and finally

$$\lim_{x\to 0} \left(-\frac{x^2}{x}\right) = \lim_{x\to 0} (-x) = 0.$$

Therefore, the trivial solution of (3.4) is uniformly exponentially asymptotically stable.

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