CHALLENGING PROBLEMS FOR CALCULUS STUDENTS

MOHAMMAD A. RAMMAHA

1. Introduction

In what follows I will post some challenging problems for students who have had some calculus, preferably at least one calculus course. All problems require a proof. They are not easy but not impossible. I hope you will find them stimulating and challenging.

2. Problems

(1) Prove that

$$e^{\pi} > \pi^{e}. \tag{2.1}$$

Hint: Take the natural log of both sides and try to define a suitable function that has the essential properties that yield inequality 2.1.

(2) Note that $\frac{1}{4} \neq \frac{1}{2}$; but $\left(\frac{1}{4}\right)^{\frac{1}{4}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$. Prove that there exists infinitely many pairs of positive real numbers α and β such that $\alpha \neq \beta$; but $\alpha^{\alpha} = \beta^{\beta}$. Also, find all such pairs.

Hint: Consider the function $f(x) = x^x$ for $x > 0$. In particular, focus your attention on the interval $(0, 1]$. Proving the existence of such pairs is fairly easy. But finding all such pairs is not so easy. Although such solution pairs are well known in the literature, here is a neat way of finding them: look at an article written by Jeff Bomberger[1], who was a freshman at UNL enrolled in my calculus courses 106 and 107, during the academic year 1991-92.

(3) Let $a_0, a_1, ..., a_n$ be real numbers with the property that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + ... + \frac{a_n}{n+1} = 0.$$

Prove that the equation

$$a_0 + a_1 x + a_2 x^2 + ... a_n x^n = 0$$

[1] Jeffrey Bomberger, On the solutions of $a^a = b^b$, *Pi Mu Epsilon Journal*, **Volume 9**(9)(1993), 571-572.

has at least one solution in the interval $(0,1)$.

(4) Suppose that f is a continuous function on $[0,2]$ such that $f(0) = f(2)$. Show that there is a real number $\xi \in [1,2]$ with $f(\xi) = f(\xi - 1)$.

(5) Suppose that $f : [0,1] \longrightarrow [0,1]$ is a continuous function. Prove that f has a fixed point in $[0,1]$, i.e., there is at least one real number $x \in [0,1]$ such that $f(x) = x$.

(6) The axes of two right circular cylinders of radius a intersect at a right angle. Find the volume of the solid of intersection of the cylinders.

(7) Let f be a real-valued function defined on $[0,\infty)$, with the properties: f is continuous on $[0,\infty)$, $f(0) = 0$, f' exists on $(0,\infty)$, and f' is monotone increasing on $(0,\infty)$.
Let g be the function given by: $g(x) = \frac{f(x)}{x}$ for $x \in (0,\infty)$.

a) Prove that g is monotone increasing on $(0,\infty)$.

b) Prove that, if $f'(c) = 0$ for some $c > 0$, and if $f(x) \geq 0$, for all $x \geq 0$, then $f(x) = 0$ on the interval $[0,c]$.

(8) Evaluate the integral $\displaystyle\int \frac{1}{x^4+1}dx$.
Hint: write $x^4 + 1$ *as* $(x^2+1)^2 - 2x^2$. *Factorize and do a partial fraction decomposition.*

(9) Determine whether the improper integral $\displaystyle\int_0^\infty \sin(x)\sin(x^2)dx$ is convergent or divergent.
Hint: the integral is convergent.

(10) Let f be a real-valued function such that f, f', and f'' are all continuous on $[0,1]$. Consider the series $\sum_{k=1}^\infty f(\frac{1}{k})$.
(a) Prove that if the series $\sum_{k=1}^\infty f(\frac{1}{k})$ is convergent, then $f(0) = 0$ and $f'(0) = 0$.
(b) Conversely, show that if $f(0) = f'(0) = 0$, then the series $\sum_{k=1}^\infty f(\frac{1}{k})$ is convergent.

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA
E-mail address: rammaha@math.unl.edu