Thick subcategories of perfect complexes over a commutative ring

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Let R a commutative noetherian ring and $\mathcal{D}$ the derived category of R-modules. A *perfect complex* of R-modules is one of the form

$$0 \to P_s \to \cdots \to P_i \to 0$$

where each P_i is a finitely generated projective R-module. Let $\mathcal{P}$ the full subcategory of $\mathcal{D}$ consisting of complexes isomorphic to perfect complexes. These are precisely the compact objects, also called small objects, in $\mathcal{D}$.

These notes are an abstract of two lectures I gave at the workshop. The main goal of the lectures was to present various proofs of a theorem of Hopkins [7] and Neeman [8], Theorem 1 below, that classifies the thick subcategories of $\mathcal{P}$, and to discuss results from [5], which is inspired by this circle of ideas.

As usual $\operatorname{Spec} R$ denotes the set of prime ideals in R with the Zariski topology; thus, the closed subsets are precisely the subsets $\operatorname{Var}(I) = \{\mathfrak{p} \supseteq I \,|\, \mathfrak{p} \in \operatorname{Spec} R\}$, where I is an ideal in R. A subset V of $\operatorname{Spec} R$ is *specialization closed* if it is a (possibly infinite) union of closed subsets; in other words, if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals such that $\mathfrak{p}$ is in V and $\mathfrak{q} \supseteq \mathfrak{p}$, then $\mathfrak{q}$ is in V.

For a prime ideal $\mathfrak{p}$, we write $k(\mathfrak{p})$ for $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, the residue field of R at $\mathfrak{p}$. The *support* of a complex of R-modules M is the set of prime ideals

$$\operatorname{Supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \,|\, k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M \not\simeq 0\}$$

In the literature, this is sometimes referred to as the *homological support*, while the word 'support' refers to the set of primes $\mathfrak{p}$ such that $M_\mathfrak{p} \not\simeq 0$; this latter set contains $\operatorname{Supp}_R M$, but is typically larger. They coincide when the R-module $\mathrm{H}(M)$ is finitely generated, in which case $\operatorname{Supp}_R M$ is a closed subset of $\operatorname{Spec} R$.

With this notation, the theorem of Hopkins and Neeman is as follows.

Theorem 1. *There is a bijection of sets*

$$\left\{\begin{matrix} \textit{Thick subcategories} \\ \textit{of } \mathcal{P} \end{matrix}\right\} \underset{\mathsf{T}}{\overset{\mathsf{S}}{\rightleftarrows}} \left\{\begin{matrix} \textit{Specialization closed} \\ \textit{subsets of } \operatorname{Spec} R \end{matrix}\right\}$$

where the maps in question are

$$\mathsf{S}(\mathcal{T}) = \bigcup_{M \in \mathcal{T}} \operatorname{Supp}_R M \quad \textit{and} \quad \mathsf{T}(V) = \{M \,|\, \operatorname{Supp}_R M \subseteq V\}$$

Proof. Note that both S and T are inclusion reversing.

It is easy to prove $\mathsf{ST}(V) = V$ when $V \subseteq \operatorname{Spec} R$ is specialization closed.

Indeed, it is clear from definitions that $\mathsf{ST}(V) \subseteq V$. Conversely, given $\mathfrak{p}$ in V, pick a set $\{x_1, \ldots, x_n\}$ which generates the ideal $\mathfrak{p}$, and let K be the Koszul complex on $\boldsymbol{x}$. It is readily verified that $\operatorname{Supp}_R K = \operatorname{Var}(\boldsymbol{x}) = \operatorname{Var}(\mathfrak{p}) \subseteq V$, so K is in $\mathsf{T}(V)$, and hence $\mathfrak{p} \in \operatorname{Var}(\mathfrak{p}) = \operatorname{Supp}_R K \subseteq \mathsf{ST}(V)$. Therefore, $V \subseteq \mathsf{ST}(V)$.

Let $\mathcal{T}$ be a thick subcategory of $\mathcal{P}$. Evidently $\mathcal{T} \subseteq \mathsf{TS}(\mathcal{T})$, so it remains to verify the reverse inclusion.

Suppose M is in $\mathsf{TS}(\mathcal{T})$, so that $\operatorname{Supp}_R M \subseteq \mathsf{S}(\mathcal{T})$. Since the R-module $\mathrm{H}(M)$ is finitely generated, $\operatorname{Supp}_R M$ is a closed subset of $\operatorname{Spec} R$, and hence it has finitely many minimal primes. Therefore, there exist complex $N_1, \dots, N_s$ in $\mathcal{T}$ such that

$$\operatorname{Supp}_R M \subseteq \bigcup_{i=1}^{s} \operatorname{Supp}_R N_i = \operatorname{Supp}_R \big(\bigoplus_{i=1}^{s} N_i\big)\,.$$

It remains to invoke Theorem 2 below. □

In what follows, given complexes M and N in (some full subcategory) of $\mathcal{D}$, we say that N *builds* M, and write $N \implies M$ if M is in the thick subcategory generated by N; when R needs to be specified, we write $N \underset{R}{\implies} M$.

Note that if $N \implies M$, then $\operatorname{Supp}_R M \supseteq \operatorname{Supp}_R N$.

Theorem 2. *If M and N in $\mathcal{P}$ are such that $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$, then $N \underset{R}{\implies} M$.*

There are (at least) three proofs of this theorem.

First proof of Theorem 2. This is due to Neeman. The basic idea is to classify the localizing subcategories of $\mathcal{D}$. These turn out to be in bijection with arbitrary subsets of $\operatorname{Spec} R$, see [8]. Hence, if $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$, then M is in the localizing subcategory of $\mathcal{D}$ generated by N. Since M and N are both in $\mathcal{P}$, they are compact objects in $\mathcal{D}$, so another result of Neeman's [9, (2.2)], implies that M is in fact in the thick subcategory generated by N. □

Second proof of Theorem 2. This is inspired by work of Dwyer and Greenlees [3]. Consider the DG algebra $\mathcal{E} = \mathbf{R}\mathrm{Hom}_R(N,N)$, the right $\mathcal{E}$-module $\mathbf{R}\mathrm{Hom}_R(N,M)$, and the following natural morphism in $\mathcal{D}$

$$\theta : \mathbf{R}\mathrm{Hom}_R(N,M) \otimes^{\mathbf{L}}_{\mathcal{E}} N \longrightarrow M\,.$$

The point is that one knows *a posteriori* that θ represents the natural morphism $\mathbf{R}\Gamma_I(M) \to (M)$, where $\mathbf{R}\Gamma_I(M)$ is local cohomology with respect to the ideal I, with $\operatorname{Supp}_R N = \operatorname{Var}(I)$. Thus, since $\operatorname{Supp}_R M$ is contained in $\operatorname{Var}(I)$, it must be that θ is an isomorphism. This can be proved directly, as follows:

An elementary calculation shows that the support of $\operatorname{cone}(\theta)$ is a subset of $\operatorname{Supp}_R M \cup \operatorname{Supp}_R N$, and hence of $\operatorname{Supp}_R N$, by hypothesis. On the other hand, $\mathbf{R}\mathrm{Hom}_R(N, \operatorname{cone}(\theta)) \simeq 0$, since $\mathbf{R}\mathrm{Hom}_R(N,\theta)$ is isomorphism, as can be easily verified, keeping in mind that N is compact. Given this, it is not difficult to prove that $\operatorname{cone}(\theta) \simeq 0$, so θ is an isomorphism.

Now, the R-algebra $\mathrm{H}(\mathcal{E})$ is noetherian, and $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(N,M))$ is finitely generated over $\mathrm{H}(\mathcal{E})$, so in the derived category of right $\mathcal{E}$-modules, one has that

$$\mathbf{R}\mathrm{Hom}_R(N,M) \simeq \operatorname*{hocolim}_n X^n$$

where X^n is in the thick subcategory generated by $\mathcal{E}$, that is to say, $\mathcal{E} \underset{\mathcal{E}}{\implies} X^n$. Therefore, in $\mathcal{D}$, one has isomorphisms

$$M \simeq \mathbf{R}\mathrm{Hom}_R(N,M) \otimes^{\mathbf{L}}_{\mathcal{E}} N \simeq \operatorname*{hocolim}_n \big(X^n \otimes^{\mathbf{L}}_{\mathcal{E}} N\big)\,,$$

where the first isomorphism is θ, and the second one is obtained by base change. Since M is compact, a standard argument yields that M is a retract of $X^n \otimes^{\mathbf{L}}_{\mathcal{E}} N$, for some N. It remains to note that by base change

$$\mathcal{E} \underset{\mathcal{E}}{\Longrightarrow} X^n \quad \text{implies} \quad N \underset{R}{\Longrightarrow} X^n \otimes^{\mathbf{L}}_{\mathcal{E}} N\,.$$

Therefore, M is in the thick subcategory generated by N, as desired. □

Third proof of Theorem 2. This proof is the original one, due to Hopkins [7], also see [8], especially the discussion on the first page, and Thomason's article [10]. The main step in it is the proof of the following 'tensor-nilpotence' theorem.

Theorem 3. *Let $\alpha : X \to Y$ be a morphism of perfect complexes. If for each $\mathfrak{p}$ in* $\operatorname{Spec} R$*, one has $\mathrm{H}(k(\mathfrak{p}) \otimes_R \alpha) = 0$, then there exists an integer $n \geq 0$ such that*

$$\alpha^n = 0\colon X^{\otimes n} \longrightarrow Y^{\otimes n}\,.$$

In my lectures, I discussed a proof of this result, and also Hopkins' argument for Theorem 2; see [7], [8], and [10]. □

Theorem 2 gives a new approach to some problems concerning descent of properties along a *local homomorphism*

$$\varphi\colon (Q, \mathfrak{q}, h) \longrightarrow (R, \mathfrak{m}, k)\,.$$

This notation means that Q and R are (commutative noetherian) local rings, with maximal ideals $\mathfrak{q}$ and $\mathfrak{m}$, residue fields h and k, and φ is a homomorphism of rings with $\varphi(\mathfrak{q}) \subseteq \mathfrak{m}$. This is the context for the rest of this write-up.

It is a classical result, found already in Cartan and Eilenberg [2], that for any R-module M, if $\operatorname{flat\,dim}_Q R$ and $\operatorname{flat\,dim}_R M$ are both finite, then so is $\operatorname{flat\,dim}_Q M$. A complex over a ring is said to have finite flat dimension if it is isomorphic, in the derived category of the ring, to a bounded complex of flat modules.

In [6], Foxby and I proved the following converse.

Theorem 4. *Let M be a perfect complex of R-modules, with* $\mathrm{H}(M) \neq 0$.
If $\operatorname{flat\,dim}_Q M$ *is finite, then* $\operatorname{flat\,dim}_Q R$ *is finite as well.*

In [5], we use Theorem 2 to give a totally different proof of this result. The argument requires the following basic facts.

(1) $\operatorname{flat\,dim}_Q R < \infty$ if and only if $\sup(h \otimes^{\mathbf{L}}_Q R) = \sup\{n \,|\, \mathrm{H}_n(h \otimes^{\mathbf{L}}_Q R) \neq 0\} < \infty$.
(2) The subcategory $\{X \in \mathcal{D}(R) \,|\, \sup(h \otimes^{\mathbf{L}}_Q X) < \infty\}$ of $\mathcal{D}(R)$ is thick.
(3) Let K be the Koszul complex on a finite set of elements in R and Y be a complex of R-modules such that the R-module $\mathrm{H}_n(Y)$ is finitely generated for each n. If $\sup(Y \otimes_Q K)$ is finite, then $\sup(Y)$ is finite.

Indeed, when $\operatorname{flat\,dim}_Q R$ is finite, it is clear that $\sup(h \otimes^{\mathbf{L}}_Q R)$ is finite. The converse is a result of André, and is proved by a standard argument: since h is the only simple Q-module, induction on length yields that $\sup(L \otimes^{\mathbf{L}}_Q R)$ is finite for any finite length Q-module L. This is the basis of an induction on the Krull dimension of L that proves that $\sup(L \otimes^{\mathbf{L}}_Q R)$ is finite for any finitely generated Q-module L, and hence that $\operatorname{flat\,dim}_Q R$ is finite. This justifies the first claim.

The second claim is a straightforward verification. As to (3), the Koszul complex on an element x of R is the mapping cone of the morphism $R \xrightarrow{x} R$, so one obtains an exact sequence of complexes

$$0 \longrightarrow Y \longrightarrow Y \otimes_Q K \longrightarrow \Sigma Y \longrightarrow 0\,.$$

The homology long exact sequence and Nakayama's lemma imply that when $\sup(Y \otimes_Q K)$ is finite, so is $\sup(Y)$, as desired. The general case is settled by an induction on the number of elements, for the corresponding Koszul complex can be realized as an iterated mapping cone.

Proof of Theorem 4. Let K be the Koszul complex on a finite set of generators for $\mathfrak{m}$. Since $\mathfrak{m}$ is the unique closed point of $\operatorname{Spec} R$, and $\operatorname{Supp}_R M$ is a closed subset of $\operatorname{Spec} R$, Theorem 2 implies that $M \implies K$. In view of (2) above, this explains the third implication in the chain below:

$$\begin{aligned}
\operatorname{flat\,dim}_Q N < \infty \quad &\implies \quad \sup(h \otimes^{\mathbf{L}}_Q N) < \infty \\
&\implies \quad \sup((h \otimes^{\mathbf{L}}_Q R) \otimes^{\mathbf{L}}_R N) < \infty \\
&\implies \quad \sup((h \otimes^{\mathbf{L}}_Q R) \otimes^{\mathbf{L}}_R K) < \infty \\
&\implies \quad \sup(h \otimes^{\mathbf{L}}_Q R) < \infty \\
&\implies \quad \operatorname{flat\,dim}_Q R < \infty\,.
\end{aligned}$$

The first implication is clear, the second one is by the associativity of tensor products, the fourth follows from (3) above applied with $Y = h \otimes^{\mathbf{L}}_Q R$, while the fifth is by (1). This completes the proof of Theorem 4. □

The paradigm of the preceding proof, see [5, (5.2)], is applicable to other homological invariants as well, and yields new results in commutative algebra, some of which are not, as yet, accessible by more 'traditional' methods.

Note that the argument above allows for a stronger conclusion: all one needs is that the thick subcategory generated by N contains a, homologically non-zero, small (i.e., compact) object; in other words, N is *virtually small*, in the terminology of [5]. This notion was suggested to us by the work in [4].

Evidently, any small object is virtually small; in [5], we identify various other classes of virtually small objects. One noteworthy result in this direction is:

Theorem 5. *Let R be a complete intersection local ring. Any complex M of R-modules with* $\mathrm{H}(M)$ *finitely generated, is virtually small.*

Compare this to the result that when R is a regular local ring, any complex M of R-modules with $\mathrm{H}(M)$ finitely generated is small. This is one direction of a theorem of Auslander, Buchsbaum, and Serre; the other direction asserts the converse. We expect that the converse to the theorem above also holds, see [5, §9].

The notion of a virtually small object carries over to any triangulated category, and the work in [4, 5] makes it is clear that it would be worthwhile to investigate such objects. It is also useful to *quantify* the process of building one object from another. This is being investigated in [1], where it provides the technical tools to bring to light an unexpected relationship between perfect complexes over a local ring and free summands of its conormal modules.

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