GRADED RINGS AND MODULES

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Throughout these notes, all rings are assumed to be commutative with identity.

§1. Definitions and examples

Definition 1.1. A ring R is called *graded* (or more precisely, $\mathbf{Z}$-*graded*) if there exists a family of subgroups $\{R_n\}_{n\in\mathbf{Z}}$ of R such that

(1) $R = \oplus_n R_n$ (as abelian groups), and
(2) $R_n \cdot R_m \subseteq R_{n+m}$ for all n, m.

A graded ring R is called *nonnegatively* graded (or $\mathbf{N}$- graded) if $R_n = 0$ for all $n \leq 0$. A non-zero element $x \in R_n$ is called a *homogeneous* element of R of degree n.

Remark 1.1. If $R = \oplus R_n$ is a graded ring, then R_0 is a subring of R, $1 \in R_0$ and R_n is an R_0-module for all n.

proof. As $R_0 \cdot R_0 \subseteq R_0$, R_0 is closed under multiplication and thus is a subring of R. To see that $1 \in R_0$, write $1 = \sum_n x_n$ where each $x_n \in R_n$ and all but finitely many of the x_n's are zero. Then for all i,

$$x_i = 1 \cdot x_i = \sum_n x_i x_n.$$

By comparing degrees, we see that $x_i = x_i x_0$ for all i. Therefore,

$$\begin{aligned} x_0 = 1 \cdot x_0 &= \sum_n x_n x_0 \\ &= \sum_n x_n = 1. \end{aligned}$$

Hence $1 = x_0 \in R_0$. The last statement follows from the fact that $R_0 \cdot R_n \subseteq R_n$ for all n.

Exercise 1.1. Prove that all units in a graded domain are homogeneous. Also, prove that if R is a graded field then R is concentrated in degree 0; i.e., $R = R_0$ and $R_n = 0$ for all $n \neq 0$.

Exercise 1.2. Let R be a graded ring and I an ideal of R_0. Prove that $IR \cap R_0 = I$.

Examples of graded rings abound. In fact, every ring R is trivially a graded ring by letting $R_0 = R$ and $R_n = 0$ for all $n \neq 0$. Other rings with more interesting gradings are given below.

1. *Polynomial rings*

Let R be a ring and $x_1, \ldots, x_d$ indeterminates over R. For $m = (m_1, \ldots, m_d) \in \mathbf{N}^d$, let $\mathbf{x}^m = x_1^{m_1} \cdots x_d^{m_d}$. Then the polynomial ring $S = R[x_1, \ldots, x_d]$ is a graded ring, where

$$S_n = \{ \sum_{m \in \mathbf{N}^d} r_m \mathbf{x}^m \mid r_m \in R \text{ and } m_1 + \cdots + m_d = n \}.$$

This is called the *standard grading* on the polynomial ring $R[x_1, \ldots, x_d]$. Notice that $S_0 = R$ and $\deg x_i = 1$ for all i. There are other useful gradings which can be put on S. Let $(\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}^d$ Then the subgroups $\{S_n\}$ where

$$S_n = \{ \sum_{m \in \mathbf{N}^d} r_m \mathbf{x}^m \mid r_m \in R \text{ and } \alpha_1 m_1 + \cdots + \alpha_d m_d = n \}$$

defines a grading on S. Here, $R \subseteq S_0$ and $\deg x_i = \alpha_i$ for all i.

As a particular example, let $S = k[x, y, z]$ (where k is a field) and $f = x^3 + yz$. Under the standard grading of S, the homogeneous components of f are x^3 and yz. However, if we give S the grading induced by setting $\deg x = 3$, $\deg y = 4$, $\deg z = 5$, then f is homogeneous of degree 9.

2. *Graded subrings*

Definition 1.2. Let $S = \oplus S_n$ be a graded ring. A subring R of S is called a *graded* subring of S if $R = \sum_n (S_n \cap R)$. Equivalently, R is graded if for every element $f \in R$ all the homogeneous components of f (as an element of S) are in R.

Exercise 1.3. Let $S = \oplus S_n$ be a graded ring and $f_1, \ldots, f_d$ homogeneous elements of S of degrees $\alpha_1, \ldots, \alpha_d$, respectively. Prove that $R = S_0[f_1, \ldots, f_d]$ is a graded subring of S, where

$$R_n = \{ \sum_{m \in \mathbf{N}^d} r_m f_1^{m_1} \cdots f_d^{m_d} \mid r_m \in S_0 \text{ and } \alpha_1 m_1 + \cdots + \alpha_d m_d = n \}.$$

Some particular examples:

(a) $k[x^2, xy, y^2]$ is a graded subring of $k[x, y]$.
(b) $k[t^3, t^4, t^5]$ is a graded subring of $k[t]$.
(c) $\mathbf{Z}[u^3, u^2 + v^3]$ is a graded subring of $\mathbf{Z}[u, v]$, where $\deg u = 3$ and $\deg v = 2$.

3. *Graded rings associated to filtrations*

Let R be a ring and $\mathcal{I} = \{I_n\}_{n=0}^{\infty}$ a sequence of ideals of R. $\mathcal{I}$ is called a *filtration* of R if

(1) $I_0 = R$,
(2) $I_n \supseteq I_{n+1}$ for all n, and
(3) $I_n \cdot I_m \subseteq I_{n+m}$ for all n, m.

Examples of filtrations are: $\{I^n\}$, where I is an ideal of R; $\{P^{(n)}\}$, where P is a prime ideal of R and $P^{(n)} = P^n R_P \cap R$ is the nth symbolic power of P; and $\{\overline{I^n}\}$, where I is an ideal of R and $\overline{I^n}$ denotes the integral closure of I^n.

Now let $\mathcal{I} = \{I_n\}$ be a filtration of R. Define the *Rees algebra* $\mathcal{R}(\mathcal{I})$ by

$$\begin{aligned} \mathcal{R}(\mathcal{I}) &= \oplus I_n \\ &= R \oplus I_1 \oplus I_2 \oplus \cdots \end{aligned}$$

where the direct sum is as R-modules and the multiplication is determined by $I_m \cdot I_n \subseteq I_{m+n}$. An alternative way to define the Rees algebra of $\mathcal{I}$ is to describe it as a subring of the graded ring $R[t]$ (where $\deg t = 1$): define

$$\mathcal{R}(\mathcal{I}) = \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in R[t] \mid a_i \in I_i \ \forall\ i\}.$$

Then $\mathcal{R}(\mathcal{I})$ is a graded subring of $R[t]$ where $\mathcal{R}(\mathcal{I})_n = \{at^n \mid a \in I_n\}$. The advantatage to this approach is that the exponent of the variable t identifies the degrees of the homogeneous components of a particular element of $\mathcal{R}(\mathcal{I})$.

Exercise 1.4. Let R be a ring, $I = (a_1, \ldots, a_k)R$ a finitely generated ideal, and $\mathcal{I} = \{I^n\}$. Prove that $\mathcal{R}(\mathcal{I}) = R[a_1 t, \ldots, a_k t]$. Generalize this statement to arbitrary ideals.

In the case $\mathcal{I} = \{I^n\}$ where I is an ideal of R, we call $\mathcal{R}(\mathcal{I})$ the Rees algebra of I and denote it by $R[It]$. By the above exercise, $R[It]$ is literally the smallest subring of $R[t]$ containing R and It. As a particular example, let $R = k[x, y]$ and $I = (x^2 + y^5, xy^4, y^6)$. Then

$$\begin{aligned} R[It] &= R[(x^2 + y^5)t, xy^4 t, y^6 t] \\ &= k[x, y, x^2 t + y^5 t, xy^4 t, y^6 t]. \end{aligned}$$

Notice in this example that in the Rees algebra grading, $\deg x = 0$, $\deg y = 0$ and $\deg t = 1$.

Another graded ring we can form with a filtration $\mathcal{I} = \{I_n\}$ of R is the associated graded ring of $\mathcal{I}$, denoted $G(\mathcal{I})$, which we now define: as an R-module,

$$\begin{aligned} G(\mathcal{I}) &= \oplus I_n / I_{n+1} \\ &= R/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots . \end{aligned}$$

To define the multiplication on $G(\mathcal{I})$, let n and m be nonnegative integers and suppose $x_n + I_{n+1}$ and $x_m + I_{m+1}$ are elements of $G(\mathcal{I})_n$ and $G(\mathcal{I})_m$, respectively. Define the product by

$$(x_n + I_{n+1})(x_m + I_{m+1}) = x_n x_m + I_{n+m+1}.$$

Exercise 1.5. Show that the multiplication defined above is well-defined.

If I is an ideal of R and $\mathcal{I} = \{I^n\}$, then $G(\mathcal{I})$ is called the associated graded ring of I and is denoted by $gr_I(R)$.

Definition 1.3. Let R be a graded ring and M an R-module. We say that M is a *graded* R-module (or has an R-grading) if there exists a family of subgroups $\{M_n\}_{n\in\mathbf{Z}}$ of M such that

(1) $M = \oplus_n M_n$ (as abelian groups), and
(2) $R_n \cdot M_m \subseteq M_{n+m}$ for all n, m.

If $u \in M \setminus \{0\}$ and $u = u_{i_1} + \cdots + u_{i_k}$ where $u_{i_j} \in R_{i_j} \setminus \{0\}$, then $u_{i_1}, \ldots, u_{i_k}$ are called the *homogeneous components* of u.

There are many examples of graded modules. As with arbitrary modules, most graded modules are constructed by considering submodules, direct sums, quotients and localizations of other graded modules. Our first observation is simply that if R is a graded ring, then R is a graded module over itself.

Exercise 1.4. Let $\{M_\lambda\}$ be a family of graded R- modules. Show that $\oplus_\lambda M_\lambda$ is a graded R-module.

Thus $R^n = R \oplus \cdots \oplus R$ (n times) is a graded R-module for any $n \geq 1$.

Given any graded R-module M, we can form a new graded R-module by twisting the grading on M as follows: if n is any integer, define $M(n)$ (read "M twisted by n") to be equal to M as an R-module, but with it's grading defined by $M(n)_k = M_{n+k}$. (For example, if $M = R(-3)$ then $1 \in M_3$.)

Exercise 1.5. Show that $M(n)$ is a graded R-module.

Thus, if $n_1, \ldots n_k$ are any integers then $R(n_1) \oplus \cdots \oplus R(n_k)$ is a graded R-module. Such modules are called *free.*

We can also obtain graded modules by localizing at a multiplicatively closed set of homogeneous elements, as illustrated in the following exercise:

Exercise 1.6. Let R be a graded ring and S a multiplicatively closed set of homogeneous elements of R. Prove that R_S is a graded ring, where

$$(R_S)_n = \{\frac{r}{s} \in R_S \mid r \text{ and } s \text{ are homogeneous and } \deg r - \deg s = n\}.$$

Likewise, prove that if M is a graded R-module then M_S is graded both as an R-module and as an R_S-module.

§2. Homogeneous ideals and submodules

Definition 2.1. Let $M = \oplus M_n$ be a graded R-module and N a submodule of M. For each $n \in \mathbf{Z}$, let $N_n = N \cap M_n$. If the family of subgroups $\{N_n\}$ makes N into a graded R-module, we say that N is a *graded* (or *homogeneous*) *submodule* of M.

Note that for any submodule N of M, $R_n \cdot N_m \subseteq N_{n+m}$. Thus, N is graded if and only if $N = \oplus_n N_n$.

Exercise 2.1. Let R and M be as above, and N an arbitrary submodule of M. Prove that $\sum_n N \cap M_n$ is a graded submodule of M.

Proposition 2.1. *Let R be a graded ring, M a graded R-module and N a submodule of M. The following statements are equivalent:*

1. *N is a graded R-module.*
2. *$N = \sum_n N \cap M_n$.*
3. *For every $u \in N$, all the homogeneous components of u are in N.*
4. *N has a homogeneous set of generators.*

proof. We prove that (4) implies (2) and leave the rest of the proof as an exercise. Let $N^* = \sum_n N \cap M_n$ and let $S = \{u_\lambda\}$ be a homogeneous set of generators for N. Note that $S \subset N^*$. Thus

$$N^* \subseteq N \subseteq \sum_\lambda Ru_\lambda \subseteq N^*.$$

In particular, an ideal of a graded ring is homogeneous (graded) if and only if it has a homogeneous set of generators. For example, if the ring $k[x, y, z]$ is given the standard grading, then $(x^2, x^3 + y^2z, y^5)$ is homogeneous, while $I = (x^2 + y^3z)$ is not. What about $(x^2z, y^3 + x^3z)$?

Exercise 2.2. Let R be a graded ring, M a graded R-module and $\{N_\lambda\}$ a collection of graded submodules of M. Prove that $\sum_\lambda N_\lambda$ and $\cap_\lambda N_\lambda$ are graded submodules of M.

Exercise 2.3. Suppose I is a homogeneous ideal of a graded ring R. Prove that $\sqrt{I}$ is homogeneous.

Exercise 2.4. Let R be a graded ring, M a graded R-module and N a graded submodule of M. Prove that $(N :_R M) = \{r \in R \mid rM \subseteq N\}$ is a homogeneous ideal of R. In particular, this shows that $\operatorname{Ann}_R M = (0 :_R M)$ is homogeneous.

Exercise 2.5. Prove that every graded ring has homogeneous prime ideals.

Proposition 2.2. *Let R be a graded ring, M a graded R-module and N a graded submodule of M. Then M/N is a graded R-module, where*

$$\begin{aligned}(M/N)_n &= (M_n + N)/N \\ &= \{m + N \mid m \in M_n\}.\end{aligned}$$

proof. Clearly, $\{(M/N)_n\}_n$ is a family of subgroups of M/N and $R_k \cdot (M/N)n = (R_k \cdot M_n + N)/N \subseteq (M_{n+k} + N)/N = (M/N)_{n+k}$. Now, if $u \in M$ and $u = \sum_n u_n$ where $u_n \in M_n$ for each n, then $u + N = \sum_n (u_n + N)$. Thus $M/N = \sum_n (M/N)_n$. Finally, suppose $\sum_n (u_n + N) = 0 + N$ in M/N, where $u_n \in M_n$ for each n. Then $\sum_n u_n \in N$ and since N is a graded submodule, $u_n \in N$ for each n. Hence $u_n + N = 0 + N$ for all n and so $M/N = \sum_n (M/N)_n$ is an internal direct sum.

Exercise 2.6. Prove that if I is a homogeneous ideal of a graded ring R then R/I is a graded ring.

Exercise 2.7. Let R be a graded ring and $N \subseteq M$ graded R-modules. Prove that $M = N$ if and only if $M_p = N_p$ for all homogeneous prime ideals p of R.

Exercise 2.8. Let R be a graded ring and M a homogeneous maximal ideal of R. Prove that $M = \cdots R_{-2} \oplus R_{-1} \oplus m \oplus R_1 \oplus R_2 \cdots$ where m is a maximal ideal of R_0.

Exercise 2.9. Let R be a nonnegatively graded ring and I_0 an ideal of R_0. Prove that $I = I_0 \oplus R_1 \oplus R_2 \oplus \cdots$ is an ideal of R. Also, show that M is a homogeneous maximal ideal of R if and only if $M = m \oplus R_1 \oplus R_2 \oplus \cdots$ for some maximal ideal m of R_0.

Exercise 2.10. Let R be a nonnegatively graded ring and $N \subseteq M$ graded R-modules. Prove that $M = N$ if and only if $M_m = N_m$ for every homogeneous maximal ideals m of R.

Exercise 2.11. Let M be a graded module and I a homogeneous ideal of R. Prove that IM is a graded submodule of M and that M/IM is a graded R/I-module.

Exercise 2.12. Let R be a nonnegatively graded ring and $M = \oplus M_n$ a graded R-module. For any integer k, let $M_{\geq k} = \oplus_{n \geq k} M_n$. Prove that $M_{\geq k}$ is a graded submodule of M. In particular, this shows that $R_+ = R_{\geq 1}$ is a homogeneous ideal of R.

Definition 2.2. Let R be a graded ring and M, N graded R-modules. Let $f\colon M \mapsto N$ be an R-module homomorphism. Then f is said to be *graded* or *homogeneous* of *degree* d if $f(M_n) \subseteq N_{n+d}$ for all n.

As an elementary example of a graded homomorphism, let M be an R-module and $r \in R_d$. Define $\mu_r : M \mapsto M$ by $\mu_r(m) = rm$ for all m in M. Then μ_r is a graded homomorphism of degree d.

Remark 2.1. If $f\colon M \mapsto N$ is a graded homomorphism of degree d, then $f\colon M(-d) \mapsto N$ is a degree 0 homomorphism.

Let M be a graded R-module. We'll construct a homogeneous map of degree 0 from a graded free R-module onto M. Let $\{m_\lambda\}$ be a homogeneous set of generators for M, where $\deg m_\lambda = n_\lambda$. For each λ, let e_λ be the unit element of $R(-n_\lambda)$. Then the R-module homorphism $f\colon \oplus_\lambda R(-n_\lambda) \mapsto M$ determined by $f(e_\lambda) = m_\lambda$ for all λ is a degree 0 map of a graded free module onto M.

Exercise 2.13. Prove that if $f\colon M \mapsto N$ is a graded homorphism of graded R-modules then $\ker(f)$ is a graded submodule of M and $\operatorname{im}(f)$ is a graded submodule of N.

Exercise 2.14. Let $C_.$ be a complex of graded R-modules with homogeneous maps. Prove that the homology modules $H_i(C_.)$ are graded for all i.

Definition 2.15. Let R and S be graded rings and $f\colon R \mapsto S$ a ring homomorphism. Then f is called a *graded* or *homogeneous* ring homorphism if $f(R_n) \subseteq S_n$ for all n.

Remark 2.2. Recall that any ring homomorphism $f\colon R \mapsto S$ induces an R-module structure on S via $r \cdot s = f(r) \cdot s$ for all $r \in R$ and $s \in S$. If R and S are graded, then f is homogeneous if and only if the grading for S is an R-module grading.

Let $R = k[x, y]$ have the standard grading (where k is a field). Then the ring homomorphism $f\colon R \mapsto R$ determined by $f(x) = x + y$ and $f(y) = x$ (i.e., $f(g(x, y)) = g(x + y, x)$) is a graded ring homomorphism, but the ring map $h\colon R \mapsto R$ defined by $h(x) = x^2$ and $h(y) = xy$ is not graded, as $h(R_n) \subset R_{2n}$. However, we can make h into a graded homomorphism as follows: let $S = k[x, y]$ where $\deg x = \deg y = 2$. Then $h\colon S \mapsto R$ as defined above is now graded.

As another example, define a ring map $f\colon k[x, y, z] \mapsto k[t^3, t^4, t^5]$ by $f(x) = t^3$, $f(y) = t^4$ and $f(z) = t^5$. If we set $\deg t = 1$, $\deg x = 3$, $\deg y = 4$ and $\deg z = 5$ then f is homogeneous.

Definition 2.4. Let R be a graded ring. We say two graded R-modules M and N are *isomorphic as graded modules* if there exists a degree 0 isomorphism from M to N. Likewise, two graded rings R and S are said to be *isomorphic as graded rings* if there exists a homogeneous ring isomorphism between them.

Exercise 2.15. Let R be a ring, I an ideal of R and $S = R[It]$. Prove that $IS \cong S_+(1)$ as graded S-modules and $S/IS \cong gr_I(R)$ as graded rings.

Exercise 2.16. Let R be a nonnegatively graded ring such that $R = R_0[R_1]$ and let $I = R_+$. Prove that $gr_I(R) \cong R$ as graded rings.

§3. Primary decompositions of graded submodules

Definition 3.1. Suppose M is a graded R-module and N an R-submodule of M. We denote by N^* the R-submodule of M generated by all the homogeneous elements contained in N. Clearly, N^* is the largest homogeneous submodule of M contained in N.

Exercise 3.1. Let M be a graded R-module and $\{N_\lambda\}_\lambda$ a family of submodules of M. Prove that $\cap N_\lambda^* = (\cap N_\lambda)^*$. Also, show that it is not necessarily true that $\sum N_\lambda^* = (\sum N_\lambda)^*$.

Exercise 3.2. Let R be a graded ring, M a graded R-module and N a submodule of M. Prove that $\sqrt{\operatorname{Ann}_R M/N^*} = (\sqrt{\operatorname{Ann}_R M/N})^*$

Theorem 3.1. *Let R be a graded ring and M a graded R-module.*

1. *If p is a prime ideal of R, so is p^*.*
2. *If N is a p-primary submodule of M then N^* is p^*-primary.*

proof. We begin by showing that if N is primary then so is N^*. By passing to the module M/N^*, we may assume $N^* = 0$. So suppose $r \in R$ is a zero-divisor on M. We need to show r is nilpotent on M. Let n be the number of (non-zero) homogeneous components of r. We'll use induction on n to show $r \in \sqrt{\operatorname{Ann}_R M}$. If $n = 0$ then $r = 0$ and there's nothing to show. Suppose $n > 0$. Let $x \in M \setminus \{0\}$ such that $rx = 0$ and let r_k and x_t be the homogeneous components of r and x (respectively) of highest degree. Then $r_k x_t = 0$ and since $N^* = 0$, $x_t \notin N$. Thus r_k is a zero-divisor on M/N. Since N is primary, r_k is nilpotent on M/N and so $r_k^e M \subseteq N$ for some integer $e \geq 1$. But since $r_k^e M_n \subseteq N$ for each n and $N^* = 0$, we conclude that $r_k^e M = 0$. Thus, $r_k \in \sqrt{\operatorname{Ann}_R M}$.

Now, choose m such that $r_k^m x = 0$ but $r_k^{m-1} x \neq 0$. Let $x' = r_k^{m-1} x$ and $r' = r - r_k$. Then $r'x' = rx' - r_k x' = 0$ and so r' is a zero-divisor on M with one less homogeneous component than r. By induction, $r' \in \sqrt{\operatorname{Ann}_R M}$ and so $r = r' + r_k \in \sqrt{\operatorname{Ann}_R M}$. This proves that N^* is primary. Moreover, if N is primary to $p = \sqrt{\operatorname{Ann}_R M/N}$ then, using Exercise 3.2, N^* is primary to $\sqrt{\operatorname{Ann}_R M/N^*} = p^*$. This completes the proof of (2).

To prove (1), apply part (2) to $M = R$ and $N = p$ and use the fact that the radical of a primary ideal is prime.

Corollary 3.1. *Let R be a graded ring and I a homogeneous ideal. Then every minimal prime over I is homogeneous. In particular, every minimal prime of the ring is homogeneous.*

proof. If $p \supseteq I$ then $p^* \supseteq I$. So if p is minimal over I then $p = p^*$.

Exercise 3.3. Let R be a graded ring and $f = \sum f_n$ (where $f_n \in R_n$) an element of R such that f_0 is not in any minimal prime of R. Prove that f is a unit if and only if f_0 is a unit in R_0 and f_n is nilpotent for all $n \neq 0$.

Exercise 3.4. Let R be a graded ring which has a unique maximal ideal. Prove that R_0 has a unique maximal ideal and every element in R_n ($n \neq 0$) is nilpotent. (Hint: first show that the maximal ideal of R is homogeneous and then apply the Exercise 3.3.

Corollary 3.2. *Let M be a graded R-module and N a graded submodule. If N has a primary decomposition, then all the primary components of N can be chosen to be homogeneous. In particular, all the isolated primary components of N and all the associated primes of M/N are homogeneous.*

proof. Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ be a primary decomposition of N. Then

$$\begin{aligned} N &= N^* \\ &= (Q_1 \cap \cdots \cap Q_k)^* \\ &= Q_1^* \cap \cdots \cap Q_k^* \end{aligned}$$

is a primary decomposition of N with homogeneous primary submodules.

Exercise 3.5. Suppose R is a Noetherian graded ring and $N \subset M$ finitely generated graded R-modules. Prove that if $p \in \operatorname{Ass}_R(M/N)$ then $p = (N :_R x)$ for some homogeneous element $x \in M \setminus N$.

§4. Noetherian and Artinian properties

Exercise 4.1. Let R be a graded ring, M a graded R-module and N an R_0-submodule of M_n for some n. Prove that $RN \cap M_n = N$.

Lemma 4.1. *Let R be a graded ring and M a Noetherian (Artinian) graded R-module. Then M_n is a Noetherian (respectively, Artinian) R_0-module for all n.*

proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of R_0-submodules of M_n. Then $RN_1 \subseteq RN_2 \subseteq RN_3 \subseteq \cdots$ is an ascending chain of R-submodules of M and so must stabilize. Contracting back to M_n and using the above exercise, we see that the chain $N_1 \subseteq N_2 \subseteq \cdots$ stabilizes. A similar argument works if M is Artinian.

Theorem 4.1. *A graded ring R is Noetherian if and only if R_0 is Noetherian and R is finitely generated (as an algebra) over R_0*

proof. If R_0 is Noetherian and R is a f.g. R_0-algebra then R is Noetherian by the Hilbert Basis Theorem.

Suppose R is Noetherian. By Lemma 4.1, R_0 must also be Noetherian. We need to prove that R is f.g. over R_0. Let $R_- = \oplus_{n<0} R_n$ and $R_+ = \oplus_{n>0} R_n$. We first show that $R_0[R_-]$ is finitely generated over R_0. If $R_- = 0$ then there is nothing to show. Otherwise, let $y_1, \ldots, y_d \in R_-$ be ideal generators for $(R_-)R$. Since R_- is a homogeneous ideal we may assume that these generators are homogeneous. Let $-k = \min\{\deg y_1, \ldots, \deg y_d\}$ ($k > 0$). Let $N = R_{-k} \oplus R_{-k+1} \oplus \cdots \oplus R_{-1}$. By the lemma, N is finitely generated as an R_0-module, so let $x_1, \ldots, x_t$ be homogeneous generators for N as an R_0-module. Clearly, $(x_1, \ldots, x_t)R = (y_1, \ldots, y_d)R = (R_-)R$.

We claim that $R_0[R_-] = R_0[x_1, \ldots, x_t]$. Let $S = R_0[R_-]$ and $T = R_0[x_1, \ldots, x_t]$. We'll show by induction on n that $S_{-n} = T_{-n}$ for all $n \geq 0$. When $n = 0$ we have that $S_0 = T_0 = R_0$, so suppose $n > 0$. Let $r \in S_{-n}$. If $n \leq k$ then $r \in N = R_0x_1 + \cdots + R_0x_t \subseteq T$. Suppose $n > k$. Since $r \in R_-R = (x_1, \ldots, x_t)R$, there exists homogeneous elements $u_1, \ldots, u_t \in R$ such that $r = \sum u_i x_i$. Therefore, $\deg u_i + \deg x_i = \deg r = -n$ for all i. Since $-n < \deg x_i < 0$ for all i, we see that $-n < \deg u_i < 0$ for all i. By the inductive hypothesis, $u_i \in T$ for all i. Hence $r = \sum u_i x_i \in T$ and hence $S_{-n} = T_{-n}$.

Let $A = R_0[R_-]$. We now claim that R is finitely generated over A. To show this, let $z_1, \ldots, z_m \in R_+$ be homogeneous ideal generators for $(R_+)R$. By using an argument similar to the one above (for $R_0[R_-]$) one can prove that $R = A[z_1, \ldots, z_m]$. Since R is f.g. over A and A is f.g. over R_0, we see that R is f.g. over R_0. This completes the proof.

Exercise 4.2. A nonnegatively graded ring R is Noetherian if and only if R_0 is Noetherian and R_+ is a f.g. ideal.

Exercise 4.3. Let R be a graded ring. Prove that R is Noetherian if and only if R satisfies the ascending chain condition on homogeneous ideals. (Hint: use that $(R_-)R$ and $(R_+)R$ are f.g. ideals.)

Exercise 4.4. Let R be a nonnegatively graded ring which has a unique homogeneous maximal ideal M. Prove that R is Noetherian if and only if R_M is Noetherian.

Exercise 4.5. Let R be a Noetherian ring and I an ideal of R. Prove that $R[It]$ and $gr_I(R)$ are Noetherian. Also, give an example of a ring R and an ideal I such that $gr_I(R)$ is Noetherian but $R[It]$ is not.

Lemma 4.2. *Let R be a graded ring and M a graded R-module. Then M is simple as an R-module if and only if M is simple as an R_0-module.*

proof. Suppose M is simple as an R-module. Then $M \cong R/n$ for some homogeneous maximal ideal n. By Exercise 2.8, $n = \cdots \oplus R_{-2} \oplus R_{-1} \oplus m \oplus R_1 \oplus R_2 \oplus \cdots$ for some maximal ideal m of R_0. Thus $M \cong R/n \cong R_0/m$ and so M is simple as an R_0-module. The converse is trivial.

If M is an R-module, we denote the length of M as an R-module by $\lambda_R(M)$ (or simply by $\lambda(M)$ if there is no ambiguity about the underlying ring.)

Lemma 4.3. *Let R be a graded ring and M a graded R-module such that $\lambda_R(M) = n$. Then there exists chain of submodules of M*

$$M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (0)$$

such that M_i/M_{i+1} is simple and M_i is graded for all i.

proof. If $n = 0, 1$ the result is trivial, so suppose $n > 1$. By induction, it is enough to show there exists a non-zero proper graded submodule of M. Let $x \in M$ be a non-zero homogeneous element. If $Rx \neq M$, we're done, so suppose $Rx = M$. Then $M \cong R/I(d)$ (as graded R-modules), where $I = (0 :_R x)$. Thus, $\lambda_R(R/I) = n$ and so R/I is Artinian. Thus, all the maximal ideals of R/I are homogeneous (since they are minimal). If the only maximal ideal of R/I is (0), then $n = \lambda(R/I) = 1$, a contradiction. Thus, there exists a non-zero homogeneous element $r \in R \setminus I$ such that $r + I$ is not a unit in R/I. Set $y = rx$ and $N = Ry$. Then N is a non-zero proper graded submodule of M.

Theorem 4.2. *Let R be a graded ring and M a graded R-module. Then*

$$\begin{aligned}\lambda_R(M) &= \lambda_{R_0}(M)\\ &= \sum_n \lambda_{R_0}(M_n).\end{aligned}$$

proof. If $\lambda_R(M) = \infty$ the $\lambda_{R_0}(M) = \infty$, so suppose $\lambda_R(M) = n$. Then by Lemma 4.3 there exists a composition series

$$M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (0)$$

where M_i/M_{i+1} are graded simple R-modules for all i. By Lemma 4.2, these modules are simple R_0-modules as well. Hence, $\lambda_{R_0}(M) = n$.

Corollary 4.1. *Let R be a graded ring and M a graded R-module. Then M has finite length as an R-module if and only if each M_n has finite length as an R_0-module and $M_n = 0$ for all but finitely many n.*

proof. Immediate from Theorem 4.2.

Corollary 4.2. *A graded ring R is Artinian if and only if R_n is an Artinian R_0-module for all n and $R_n = 0$ for all but finitely many n.*

proof. A ring is Artinian if and only if it has finite length.

We remark that this corollary does not hold for modules, as the following example illustrates:

Example 4.1. Let k be a field and $R = k[x]$ where x is an indeterminate of degree 1. Then $R_x = k[x^{-1}, x]$ and R is a graded R-submodule of R_x. Let $M = R_x/R$. We claim that M is an Artinian R-module and $M_n \neq 0$ for all $n < 0$.

Note that

$$\begin{aligned} M &= k[x^{-1}, x]/k[x] \\ &= \cdots \oplus kx^{-n} \oplus kx^{-n+1} \oplus \cdots \oplus kx^{-1} \end{aligned}$$

Hence, $M_n \neq 0$ for all $n < 0$. It is easy to see that if N is a submodule of M and $a_{-n}x^{-n} + \cdots + a_{-1}x^{-1} \in N$ where $a_{-n} \neq 0$ then $\{x^{-n}, \ldots, x^{-1}\} \subset N$. Hence, every proper submodule of M is of the form $kx^{-n} \oplus \cdots \oplus kx^{-1}$ for some n. Thus M is Artinian (but not Noetherian).

Exercise 4.6. Let k be a field and $R = k[x, y, z]/(x^2, y^2, z^3)$. Find $\lambda(R)$.

Exercise 4.7. Let R be a nonnegatively graded ring. Prove that R is Artinian if and only if R satisfies the descending chain condition on homogeneous ideals. Is this true for $\mathbf{Z}$-graded rings?

Exercise 4.8. Let R be a nonnegatively Noetherian graded ring such that R_0 is Artinian and R_+ is a nilpotent ideal. Prove that R is Artinian. Give an example to show this is false if the Noetherian hypothesis is removed.

Exercise 4.9. Let R be a nonnegatively graded ring such that R_0 has a unique maximal ideal. Let M be the unique homogeneous maximal ideal of R. Prove that R is Artinian if and only if R_M is Artinian.

Exercise 4.10. Let R be a nonnegatively graded ring and M an Artinian graded R-module. Prove that $M_n = 0$ for all n sufficiently large.

§5. Height and dimension in graded rings

Exercise 5.1. Let R be a reduced graded ring where R_0 is a field and let $u \in R_n \setminus \{0\}$ with $n \neq 0$. Prove that u is transcendental over R_0.

Lemma 5.1. *Let R be a graded ring which is not a field and suppose the only homogeneous ideals of R are (0) and R. Then $R = k[t^{-1}, t]$ where $k = R_0$ is a field and t is a homogeneous element of R transcendental over k.*

proof. Since every non-zero homogeneous element of R is a unit, all the non-zero elements of R_0 are units and so R_0 is a field. As R is not a field, there exist some $t \in R_n$ $(n \neq 0)$

such that $t \neq 0$. Since t is a unit, $t^{-1} \in R_{-n}$ and so without loss of generality, we may assume that n is the smallest positive integer such that $R_n \neq 0$. By Exercise 5.1, we know t is transcendental over R_0.

We'll show that every homogeneous element in R_m is of the form ct^i for some i. This is trivially true when $0 \leq m < n$. So suppose $m \geq n$ and let $u \in R_m$. Then $t^{-1}u \in R_{m-n}$ and $0 \leq m - n < m$, so by induction $t^{-1}u = ct^i$ for some i. Multiplying both sides by t, we're done. A similar argument works for homogeneous elements of negative degrees. Thus $R = R_0[t^{-1}, t]$.

Lemma 5.2. *Let R be a graded ring and P a non-homogeneous prime ideal of R. Then there are no prime ideals properly between P and P^*.*

proof. By passing to R/P^* we may assume that R is a domain and that $P^* = 0$. Let W be the set of all non-zero homogeneous elements of R. Since $P \cap W = \emptyset$, PR_W is a non-zero prime ideal of R_W. Since every non-zero homogeneous element of R_W is a unit, we have by the above lemma that $R_W = k[t^{-1}, t]$. Since $\dim k[t^{-1}, t] = 1$ there are no primes properly between (0) and PR_W. Hence, there are no primes of R properly between (0) and P.

Theorem 5.1. *(Matijevic-Roberts) Let R be a graded ring and P a non-homogeneous prime ideal of R. Then* $\operatorname{ht}(P) = \operatorname{ht}(P^*) + 1$.

proof. If $\operatorname{ht}(P^*) = \infty$ then the result is trivial, so assume $\operatorname{ht}(P^*) < \infty$. We'll use induction on $n = \operatorname{ht}(P^*)$. If $n = 0$ we are done by Lemma 5.2. Suppose $n > 0$ and let Q be any prime ideal properly contained in P. It suffices to show that $\operatorname{ht}(Q) \leq n$. Now $Q^* \subseteq P^*$. If $Q^* = P^*$ then $Q = P^*$ (by Lemma 5.2) and we're done. If $Q^* \neq P^*$ then $\operatorname{ht}(Q^*) \leq n - 1$. Hence $\operatorname{ht}(Q) \leq n$ by induction.

Corollary 5.1. *Let R be a graded ring and M a finitely generated graded R-module. Let $p \in \operatorname{Supp} M$, where p is a not homogeneous. Then* $\dim M_p = \dim M_{p^*} + 1$.

proof. By passing to $R/\operatorname{Ann}_R M$ we may assume $\operatorname{Ann}_R M = 0$. Thus, $\dim M_p = \dim R_p = \operatorname{ht}(p)$ for any $p \in \operatorname{Supp} M$. The result now follows from Theorem 5.1.

Corollary 5.2. *Let R be a nonnegatively graded ring. Then* $\dim R = \max\{\operatorname{ht}(M) \mid M$ *a homogeneous maximal ideal*$\}$.

proof. Let N be a maximal ideal of R. Then $\operatorname{ht}(N^*) = \operatorname{ht}(N^*) - 1$ by the Theorem 5.1. Since N^* is homogeneous and R is nonnegatively graded, N^* is contained in a homogeneous maximal ideal M (see Exercise 2.9). Since $M \neq N^*$, $\operatorname{ht}(M) \geq \operatorname{ht}(N^*) + 1 = \operatorname{ht}(N)$.

Proposition 5.1. *Let R be a Noetherian graded ring and P a homogeneous prime ideal of height n. Then there exists a chain of distinct homogeneous prime ideals*

$$P_0 \subset P_1 \subset P_2 \cdots \subset P_n = P.$$

proof. The result is trivially true if $n = 0$ so assume $n > 0$. Let Q be a prime ideal contained in P such that $\operatorname{ht}(Q) = n - 1$. If Q is homogeneous we're done by induction, so suppose Q is not homogeneous. Then $\operatorname{ht}(Q^*) = n - 2$. By passing to R/Q^* we can assume R is a graded domain and P is a homogeneous prime of height two. It is enough to show there exist a homogeneous prime ideal of height one contained in P. Let $f \in P$ be a non-zero homogeneous element of P. Then P is not minimal over (f) by KPIT, so let P_1 be a prime contained in P which contains (f). Then P_1 is minimal over (f) (as $\mathrm{ht}(P) = 2$) and hence homogeneous.

Corollary 5.2. *Let R be a nonnegatively graded Noetherian ring. Then*

$$\dim R = \sup_n \{P_0 \subset P_1 \subset \cdots \subset P_n \mid P_0, \ldots, P_n \text{ are homogeneous primes of } R\}.$$

proof. This follows from Corollary 5.2 and Proposition 5.1.

Exercise 5.2. Let R be a Noetherian graded ring and I an ideal of R. Prove that $\mathrm{ht}(I) - 1 \leq \mathrm{ht}(I^*) \leq \mathrm{ht}(I)$.

The following result can be found in Appendix V of Zariski-Samuel, Vol II:

Proposition 5.2. *(Graded version of prime avoidance)Let R be a graded ring and I a homogeneous ideal generated by homogeneous elements of positive degree. Suppose $P_1, \ldots, P_n$ are homogeneous prime ideals, none of which contain I. Then there exists a homogeneous element $x \in I$ with $x \notin P_i$ for all i.*

proof. Without loss of generality, we may assume there are no containment relations among the ideals $P_1, \ldots, P_n$. Thus for each i, P_i does not contain the homogeneous ideal $P_1 \cap \cdots \cap \hat{P_i} \cdots \cap P_n$. Hence, there exists a homogeneous element $u_i \notin P_i$ such that $u_i \in P_j$ for all $j \neq i$. Also, for each i there exists a homogeneous element $w_i \in I \setminus P_i$ of positive degree. By replacing w_i by a sufficiently large power of w_i, we may assume that $\deg u_i w_i > 0$. Let $y_i = u_i w_i$. Then $y_i \notin P_i$ but $y_i \in I \cap P_1 \cap \cdots \cap \hat{P_i} \cap \cdots \cap P_n$. By taking powers of y_i, if necessary, we can assume that $\deg y_i = \deg y_j$ for all i, j. Now let $x = y_1 + \cdots + y_n$. Then x is homogeneous, $x \in I$ and $x \notin P_i$ for all i.

Remark 5.1. We note that Proposition 5.2 is false without the assumption that I is generated by elements of positive degree. For example, let $S = \mathbf{Z}_{(2)}$ and $R = S[x]$ where $\deg x = 1$. Let $I = (2, x)R$, $P_1 = (2)R$ and $P_2 = (x)R$. Then there does not exist a homogeneous element $x \in I$ such that $x \notin P_1 \cup P_2$.

Lemma 5.3. *Let (R, m) be a local ring such that R/m is infinite. Let M be an R-module and $Q, N_1, \ldots, N_s$ submodules of M such that Q is not contained in any N_i. Then there exists $x \in Q$ such that $x \notin N_i$ for $i = 1, \ldots, s$.*

proof. Suppose by way of contradiction that $Q \subseteq \cup_i N_i$. For each $i = 1, \ldots, s$ let $x_i \in Q \setminus N_i$. By replacing Q with $Rx_1 + \cdots + Rx_s$ and N_i with $N_i \cap Q$, we may assume Q is finitely generated and $Q = \cup_i N_i$. Then

$$Q/mQ = \bigcup_i (N_i + mQ)/mQ.$$

Since a vector space over an infinite field is not the union of a finite number of proper subspaces, we must have that $Q/mQ = (N_i + mQ)/mQ$ for some i. By Nakayama's lemma $Q = N_i$, a contradiction.

Exercise 5.4. Let R be a nonnegatively graded ring such that R_0 is local with infinite residue field. Let $I, J_1 \ldots, J_s$ be homogeneous ideals of R such that I is not contained in any J_i. Prove that there exists $u \in I$ such that $u \notin J_i$ for all i.

The following corollary allows us in certain situations to avoid ideals which are not even prime or homogeneous:

Corollary 5.3. *Let R be a graded ring such that R_0 is local with infinite residue field. Let I be an ideal of R generated by homogeneous elements of the same degree s. Suppose $J_1, \ldots, J_n$ are ideals of R, none of which contain I. Then there exists a homogeneous element $x \in I$ of degree s such that $x \notin J_i$ for all i.*

proof. Clearly $I \cap R_s$ is not contained in $J_i \cap R_s$ for any i, else $I \subset J_i$. Applying Lemma 5.3, there exists $x \in I \cap R_s$ such that $x \notin J_i$ for all i.

Exercise 5.5. Give an example to show Corollary 5.3 may be false if the residue field of R_0 is not infinite.

Theorem 5.2. *Let R be a Noetherian graded ring and P a homogeneous prime ideal of height n. Suppose that either*

(a) *P is generated by elements of positive degree, or*
(b) *R_0 is local with infinite residue field and P is generated by elements of the same degree s.*

Then there exist homogeneous elements $w_1, \ldots, w_n \in P$ such that P is minimal over $(w_1, \ldots, w_n)R$. Moreover, in case (b) we may choose $w_1, \ldots, w_n$ in degree s.

proof. If $n = 0$ the result is trivial, so we may assume that $n > 0$. Let $Q_1, \ldots, Q_n$ be the minimal primes of R. Since P is not contained in any Q_i, there exists a homogenous element $w_1 \in P$ (of degree s in case (b)) such that $w_1 \notin Q_i$ for all i. Then $\operatorname{ht}(P/(w_1)) = n - 1$. The result now follows by induction.

Corollary 5.4. *Let R be a Noetherian nonnegatively graded ring with R_0 Artinian and local. Let M be the homogeneous maximal ideal and $d = \dim R = \operatorname{ht}(M)$. Then there exist homogeneous elements $w_1, \ldots w_d \in R_+$ such that $M = \sqrt{(w_1, \ldots, w_d)}$. If in addition $R = R_0[R_1]$ and the residue field of R_0 is infinite, we can choose $w_1, \ldots, w_d$ to be in R_1.*

proof. Let m be the maximal ideal of R_0. Since mR is nilpotent, we can pass to the ring R/mR and assume that R_0 is a field; i.e., $M = R_+$. By Theorem 5.2, there exists homogeneous elements $w_1, \ldots, w_d$ in R_+ (or in R_1 if the additional hypotheses are satisfied) such that M is minimal over $(w_1, \ldots, w_d)$. As every prime minimal over $(w_1, \ldots w_d)$ is homogeneous and hence contained in M, M is the only prime containing $(w_1, \ldots, w_d)$. Thus, $M = \sqrt{(w_1, \ldots, w_d)}$.

Remark 5.2. We note that Corollary 5.4 is false if the hypothesis that R_0 is Artinian is removed. For example, let $R = \mathbf{Z}_{(2)}[x]/(2x)$. Then $\dim R = 1$ and $M = (2, x)R$, but there does not exist a homogeneous element $w \in R$ such that $M = \sqrt{(w)}$.

Exercise 6.1. Let $A \subseteq B$ be rings and W a multiplicatively closed subset of A. Assume that no element of W is a zero-divisor in B, so that we may consider B as a subring of B_W. Suppose $f \in B_W$ is integral over A_W. Prove that there exists $w \in W$ such that wf is in B and integral over A.

Exercise 6.2. Let $A \subset B$ be rings and x an indeterminate over B. Then a polynomial $f(x) \in B[x]$ is integral over $A[x]$ if and only if all the coefficients of $f(x)$ are integral over A. (Hint: see Atiyah-Macdonald, page 68, exercise 9.)

Theorem 6.1. *(Bourbaki) Let R be a graded subring of the graded ring S and let T be the integral closure of R in S. Then T is a graded subring of S.*

proof. Let t be an indeterminant over S. Define a ring homomorphism $f\colon S \mapsto S[t^{-1}, t]$ as follows: for $s = \sum_n s_n \in S$ let $f(s) = \sum_n s_n t^n$. Now, suppose $s = \sum_n s_n \in S$ is integral over R. We need to show each s_n is integral over R. As f is a ring homomorphism and $f(R) \subset R[t^{-1}, t]$, we see that $f(s)$ is integral over $R[t^{-1}, t]$. Let W be the multiplicatively closed subset $\{t^n\}_{n \geq 0}$ of $R[t]$. Then $S[t^{-1}, t] = S[t]_W$ and $R[t^{-1}, t] = R[t]_W$. By Exercise 6.1, there exists $t^n \in W$ such that $t^n f(s)$ is in $S[t]$ and integral over $R[t]$. By Exercise 6.2, this means that all the coefficients of $t^n f(s)$ (which are the s_n's) are integral over R.

Theorem 6.2. *Let $R = R_0[R_1]$ be a Noetherian graded ring and $w_1, \ldots, w_d \in R_1$. The following statements are equivalent:*

(1) $R_+ \subseteq \sqrt{(w_1, \ldots, w_d)}$.
(2) $(R_+)^{n+1} = (w_1, \ldots, w_d)(R_+)^n$ *for some* $n \geq 0$.
(3) R *is integral over* $R_0[w_1, \ldots, w_d]$

proof. (1) $\Rightarrow$ (2) : As R_+ is finitely generated, $(R_+)^{n+1} \subseteq (w_1, \ldots w_d)$ for some n. Let $f \in R_{n+1}$. Then $f = r_1 w_1 + \cdots + r_d w_d$ for some $r_i \in R_n$. Thus, $f \in (w_1, \ldots, w_d) R_n$ and so $R_{n+1} \subseteq (w_1, \ldots, w_d)(R_n)$. Since $R_+ = R_1 R$, $(R_+)^m = R_m R$ for any $m \geq 0$. Thus, $(R_+)^{n+1} \subseteq (w_1, \ldots, w_d)(R_+)^n$.

(2) $\Rightarrow$ (3) : Since $R = R_0[R_1]$, it is enough to show that any element $u \in R_1$ is integral over $R_0[w_1, \ldots, w_d]$. By (2), $uR_n \subseteq R_{n+1} = R_n w_1 + \cdots + R_n w_d$ for some $n \geq 0$. Since R is Noetherian, R_n is finitely generated as an R_0-module, so let $f_1, \ldots, f_t$ be R_0-generators for R_n. Then for each i, $uf_i = \sum_j r_{ij} f_j$ where $r_{ij} \in R_0 w_1 + \cdots + R_0 w_d$ for all i, j. If we set

$$A = \begin{pmatrix} u - r_{11} & -r_{12} & \ldots & -r_{1t} \\ -r_{21} & u - r_{22} & \ldots & -r_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ -r_{t1} & -r_{t2} & \ldots & u - r_{tt} \end{pmatrix}$$

then

$$A \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{pmatrix} = 0.$$

Multiplying by both sides by adj(A), we get that $\det(A)R_n = 0$. Since $u^n \in R_n$ we see that $\det(A)u^n = 0$. This equation shows that u is integral over $R_0[w_1, \ldots, w_d]$.

(3) $\Rightarrow$ (1) : It suffices to show that $R_1 \subset \sqrt{(w_1, \ldots, w_d)}$. So let $u \in R_1$. As u in integral over $R_0[w_1, \ldots, w_d]$, there exists an equation of the form

$$u^n + r_1 u^{n-1} + \cdots + r_n = 0$$

where $r_1, \ldots, r_n \in R_0[w_1, \ldots, w_d]$. Furthermore, since u is homogeneous of degree 1, we may assume each r_i is homogeneous of degree i. Thus $r_1, \ldots, r_n \in (w_1, \ldots, w_d)R$ and so $u^n \in (w_1, \ldots, w_d)R$. Hence $R_1 \subset \sqrt{(w_1, \ldots, w_d)}$.

Corollary 6.1. *(Graded version of Noether's normalization lemma) Let $R = R_0[R_1]$ be a d-dimensional Noetherian graded ring such that R_0 is an Artinian local ring and the residue field of R_0 is infinite. Then there exists $T_1, \ldots, T_d \in R_1$ such that R is integral over $R_0[T_1, \ldots, T_d]$. Moreover, if R_0 is a field then $R_0[T_1, \ldots, T_d]$ is isomorphic to a polynomial ring in d variables over R_0.*

proof. By Corollary 5.4, there exists $T_1, \ldots, T_d \in R_1$ such that $R_+ \subseteq \sqrt{(T_1, \ldots, T_d)}$. Hence, by the above theorem, R is integral over $R_0[T_1, \ldots, T_d]$. The last statement follows from the fact that $\dim R_0[T_1, \ldots, T_d] = \dim R = d$.

§7. Hilbert functions

A graded R module M is said to be *bounded below* if there exists $k \in \mathbf{Z}$ such that $M_n = 0$ for all $n \leq k$.

Definition 7.1. Let R be a graded ring and M a graded R-module. Suppose that $\lambda_{R_0}(M_n) < \infty$ for all n Define the *Hilbert function* $H_M : \mathbf{Z} \mapsto \mathbf{Z}$ of M by

$$H_M(n) = \lambda_{R_0}(M_n)$$

for all $n \in \mathbf{Z}$. If in addition M is bounded below, we define *Poincaré series* (or *Hilbert series*) of M to be

$$P_M(t) = \sum_{n \in \mathbf{Z}} H_M(n) t^n$$

as an element of $\mathbf{Z}((t))$.

If M is a graded R-module such that $\lambda_{R_0}(M_n) < \infty$ for all n, we say that M "has a Hilbert function" or that the Hilbert function of M "is defined." Similarly, if M is bounded below and has a Hilbert function, we say that M "has a Poincaré series." The most important class of graded modules which have Hilbert functions are those which are finitely generated over a graded ring R, where R is Noetherian and R_0 is Artinian. On the other hand, if M is a f.g. graded R-module which has a Hilbert function, then $R_0/\operatorname{Ann}_{R_0} M$ is Artinian. In fact:

Exercise 7.1. Let R be a graded ring and M a f.g. graded R-module which has a Hilbert function. Then $R/\operatorname{Ann}_R M$ has a Hilbert function.

The most important class of graded modules which have Poincaré series are those that are finitely generated over a nonnegatively graded Noetherian ring in which R_0 is Artinian. Conversely, if R is a graded ring and M is a f.g. graded R-module which has a Poincaré series, then $R/\operatorname{Ann}_R(M)$ is bounded below. Moreover, we have the following:

Exercise 7.2. Let R be a graded ring and M a finitely generated graded R-module which has a Poincaré series. Prove that $R/\operatorname{Ann}_R(M)$ has a Poincaré series and that $R/\sqrt{\operatorname{Ann}_R M}$ is nonnegatively graded.

Exercise 7.3. Let R be a graded ring and

$$0 \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$$

an exact sequence of graded R-modules with degree 0 maps. If each M_i has a Poincaré series, prove that $\sum_i (-1)^i P_{M_i}(t) = 0$.

The following Proposition gives an example of a Hilbert function which, although very simple, provides an important prototype for all Hilbert functions.

Proposition 7.1. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and $\deg x_i = 1$ for $i = 1, \ldots, d$. Then*

$$H_R(n) = \binom{n+d-1}{d-1}$$

for all $n \geq 0$.

proof. We use induction on $n + d$. The result is obvious if $n = 0$ or $d = 1$, so suppose $n > 0$ and $d > 1$. Let $S = k[x_1, \ldots, x_{d-1}]$ and consider the exact sequence

$$0 \longrightarrow R_{n-1} \xrightarrow{x_d} R_n \longrightarrow S_n \longrightarrow 0.$$

Then

$$\begin{aligned} H_R(n) = \dim_k R_n &= \dim_k R_{n-1} + \dim_k S_n \\ &= \binom{n+d-2}{d-1} + \binom{n+d-2}{d-2} \\ &= \binom{n+d-1}{d-1}. \end{aligned}$$

Theorem 7.1. *Let R be a Noetherian graded ring and M a finitely generated graded R-module which has a Poincaré series. Then $P_M(t)$ is a rational function in t. In particular, if $R = R_0[x_1, \dots, x_k]$ where $\deg x_i = s_i \neq 0$ then*

$$P_M(t) = \frac{g(t)}{\prod_{i=1}^{k}(1 - t^{s_i})}$$

where $g(t) \in \mathbf{Z}[t^{-1}, t]$.

proof. If $k = 0$ then $R = R_0$. As M is finitely generated, this means that $M_n = 0$ for all but finitely many n. Thus, $P_M(t) \in \mathbf{Z}[t^{-1}, t]$. Suppose now that $k > 0$. Then consider the exact seqence

$$0 \longrightarrow (0 :_M x_k)(-s_k) \longrightarrow M(-s_k) \xrightarrow{x_k} M \longrightarrow M/x_kM \longrightarrow 0.$$

For each n, we have that

$$\lambda(M_n)t^n - \lambda(M_{n-s_k})t^n = \lambda((M/x_kM)_n)t^n - \lambda((0 :_M x_k)_{n-s_k})t^n.$$

Summing these equations over all $n \in \mathbf{Z}$, we obtain

$$P_M(t) - t^{s_k}P_M(t) = P_{M/x_kM}(t) - t^{s_k}P_{(0:_M x_k)}(t).$$

Now, as $x_kM/x_kM = 0$ and $x_k(0 :_M x_k) = 0$, M/x_kM and $(0 :_M x_k)$ are modules over $R_0[x_1, \dots, x_{k-1}]$. As M is bounded below, so are M/x_kM and $(0 :_M x_k)$. By induction, $P_{M/x_kM}(t)$ and $P_{(0:_M x_k)}$ are of the required form, and so there exists $g_1(t), g_2(t) \in \mathbf{Z}[t^{-1}, t]$ such that

$$(1 - t^{s_k})P_M(t) = \frac{g_1(t)}{\prod_{i=1}^{k-1}(1 - t^{s_i})} - \frac{t^{s_k}g_2(t)}{\prod_{i=1}^{k-1}(1 - t^{s_i})}.$$

Dividing by $(1 - t^{s_k})$, we obtain the desired result.

An important special case is given by the following corollary:

Corollary 7.1. *Let $R = R_0[x_1, \dots, x_k]$ be a Noetherian graded ring where R_0 is Artinian and $\deg x_i = 1$ for all i. Let M be a non-zero f.g. graded R-module. Then there exists a unique integer $s = s(M)$ with $0 \leq s \leq k$ such that*

$$P_M(t) = \frac{g(t)}{(1 - t)^s}$$

for some $g(t) \in \mathbf{Z}[t^{-1}, t]$ with $g(1) \neq 0$.

proof. By Theorem 7.1, we have that $P_M(t) = \frac{f(t)}{(1-t)^k}$ for some $f(t) \in \mathbf{Z}[t^{-1}, t]$. We can write $f(t) = (1 - t)^m g(t)$ where $m \geq 0$ and $g(1) \neq 0$. If we let $s = k - m$ then we are done provided $s \geq 0$. But if $s < 0$ then $P_M(t) \in \mathbf{Z}[t^{-1}, t]$ and $P_M(1) = 0$. Hence $\sum_n \lambda(M_n) = 0$ and so $M = 0$, contrary to our assumption. The uniqueness of s and $g(t)$ is clear.

Exercise 7.4. Let R be a graded ring and M a graded R-module which has a Poincaré series. Let $x \in R_k$ ($k \neq 0$). Prove that $P_{M/xM}(t) = (1-t^k)P_M(t)$ if and only if x is not a zero-divisor on M.

Exercise 7.5. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field with $\deg x_i = k_i > 0$ for $i = 1, \ldots, d$. Prove that

$$P_R(t) = \frac{1}{\prod_{i=1}^{d}(1-t^{k_i})}.$$

Exercise 7.6. Prove that for any integer $d \geq 1$

$$\frac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \binom{n+d-1}{d-1} t^n.$$

Let k be a positive integer. Define a polynomial $\binom{x}{k} \in \mathbf{Q}[x]$ by

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

Further, define $\binom{x}{0} = 1$ and $\binom{x}{-1} = 0$. Thus, $\deg \binom{x}{k} = k$. (We adopt the convention that the degree of the zero polynomial is -1.)

Proposition 7.2. *Let M be a graded R-module having a Poincaré series of the form*

$$P_M(t) = \frac{f(t)}{(1-t)^s}$$

for some $s \geq 0$ and $f(t) \in \mathbf{Z}[t^{-1}, t]$ with $f(1) \neq 0$. Then there exists a unique polynomial $Q_M(x) \in \mathbf{Q}[x]$ of degree $s-1$ such that $H_M(n) = Q_M(n)$ for all sufficiently large integers n.

proof. Let $f(t) = a_l t^l + a_{l+1}t^{l+1} + \cdots + a_m t^m$. By exercise 7.6,

$$\begin{aligned} P_M(t) &= \frac{f(t)}{(1-t)^s} \\ &= f(t) \cdot \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} t^n. \end{aligned}$$

Comparing coefficients of t^n, we see that for $n \geq m$

$$H_M(n) = \sum_{i=l}^{m} a_i \binom{n+s-i-1}{s-1}.$$

Let $Q_M(x) = \sum_i a_i \binom{x+s-i-1}{s-1}$. Then $Q_M(x)$ is a polynomial $\mathbf{Q}[x]$ of degree at most $s-1$ and $Q_M(n) = H_M(n)$ for all n sufficiently large. Note that the coefficient of x^{s-1} is $(a_l + \cdots + a_m)/(s-1)! = f(1)/(s-1)! \neq 0$. Thus $\deg Q_M(x) = s-1$.

Definition 7.2. Let R be a graded ring and M a graded R-module which has a Hilbert function $H_M(n)$. A polynomial $Q_M(x) \in \mathbf{Q}[x]$ is called the *Hilbert polynomial* of M if $Q_M(n) = H_M(n)$ for all sufficiently large integers n

Corollary 7.2. *Let $R = R_0[x_1, \ldots, x_k]$ be a Noetherian graded ring where R_0 is Artinian and $\deg x_i = 1$ for all i. Let M be a non-zero finitely generated graded R-module. Then M has a Hilbert polynomial $Q_M(x)$ and $\deg Q_M(x) = s(M) - 1 \leq k - 1$.*

proof. Immediate from Corollary 7.1 and Proposition 7.2.

Exercise 7.7. Let k be a field and $R = k[x, y, z]/(x^3 - y^2 z)$ where $\deg x = \deg y = \deg z = 1$. Find $P_R(t)$ and $Q_R(x)$.

Exercise 7.8. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field with $\deg x_i = 1$ for all i. Suppose $f_1, \ldots, f_d$ are homogeneous elements in R_+ which form a regular sequence in R. Prove that

$$\lambda(R/(f_1, \ldots, f_d)) = \prod_{i=1}^{d} \deg f_i.$$

Definition 7.3. Let R be a graded ring and M a graded R-module. An element $x \in R_\ell$ is said to be *superficial (of order ℓ) for M* if $(0 :_M x)_n = 0$ for all but finitely many n.

Lemma 7.1. *Let R be a nonnegatively graded ring which is finitely generated as an R_0-algebra and M a finitely generated graded R-module. Then for any $k \geq 1$, $M_n \subseteq (R_+)^k M$ for n sufficiently large.*

proof. First note that if $R_n = 0$ for n sufficiently large then $M_n = 0$ for n sufficiently large. Now $N = M/(R_+)^k M$ is a finitely generated $S = R/(R_+)^k$-module. If $S_n = 0$ for $n >> 0$ then $N_n = 0$ for $n >> 0$ and we're done. Hence, it suffices to prove the Lemma in the case $M = R$. Let $R = R_0[y_1, \ldots, y_s]$ where $\deg y_i = d_i > 0$ for $i = 1, \ldots, s$. Let $p = k(d_1 + \cdots + d_s)$ and suppose $u \in R_n$ for some $n \geq p$. Then u is a finite sum of elements of the form $r y_1^{a_1} \cdots y_s^{a_s}$, where $r \in R$ and $a_1 d_1 + \cdots + a_s d_s = n$. Since $n \geq k(d_1 + \cdots + d_s)$, we must have that $a_i \geq k$ for some i. Hence, $r y_1^{a_1} \cdots y_s^{a_s} \in (R_+)^k$.

The following Proposition generalizes a result in Zariski-Samuel, Vol II:

Proposition 7.3. *Let R be a nonnegatively graded Noetherian ring and M a finitely generated graded R-module. Then there exists a homogeneous element $x \in R_+$ such that x is superficial for M. Moreover, if $R = R_0[R_1]$ and the residue field of R_0 is infinite, then we may choose $x \in R_1$.*

proof. Let

$$0 = Q_1 \cap Q_2 \cap \cdots \cap Q_t$$

be a primary decomposition of 0 in M. Let $P_i = \sqrt{\operatorname{Ann}_R M/Q_i}$ be the prime ideal associated to Q_i. We can arrange the Q_i's so that $R_+ \not\subset P_i$ for $i = 1, \ldots, s$ and $R_+ \subseteq P_i$ for $i = s+1, \ldots, t$, where $0 \leq s \leq t$.

Since $R_+ \not\subset P_1 \cup \cdots \cup P_s$, there is a homogeneous element $x \in R_+$ which is not in $P_1 \cup \cdots \cup P_s$. Let $\ell = \deg x$. If the residue field of R_0 is infinite and $R = R_0[R_1]$, then we may choose $\ell = 1$ by Corollary 5.3.

We claim that x is superficial for M. As

$$R_+ \subseteq P_{s+1} \cap \cdots \cap P_t$$

there exists $k \in \mathbf{N}$ such that

$$(R_+)^k \subseteq \operatorname{Ann}_R M/Q_{s+1} \cap \cdots \cap \operatorname{Ann}_R M/Q_t.$$

Therefore,

$$(R_+)^k M \subseteq Q_{s+1} \cap \cdots \cap Q_t.$$

By Lemma 7.1, there exists $N \in \mathbf{N}$ such that $M_n \subseteq (R_+)^k M$ for $n \geq N$. Thus, for $n \geq N$

$$M_n \subseteq Q_{s+1} \cap \cdots \cap Q_t.$$

Now suppose $u \in (0 :_M x)_n$ where $n \geq N$. Then $u \in Q_i$ for $i = s+1, \ldots, t$. But since $xu \in Q_i$ and $x \notin P_i$ for $i = 1, \ldots, s$, we see that $u \in Q_1 \cap \cdots \cap Q_s$. Hence

$$u \in Q_1 \cap \cdots \cap Q_t = 0.$$

Lemma 7.2. *Let R be a nonnegatively graded Noetherian ring and M a finitely generated graded R-module such that* $\dim M > \dim R_0$. *Then for any superficial element $x \in R_+$ for M*

$$\dim M/xM = \dim M - 1.$$

proof. Without loss of generality, we may assume that $\operatorname{Ann}_R M = 0$. Then $\dim M/xM = \dim R/xR$. Since R is nonnegatively graded, $\dim R = \dim R_N$ for some homogeneous maximal ideal N of R by Corollary 5.1. Since $x \in N$,

$$\dim R/xR \geq \dim R_N/xR_N \geq \dim R_N - 1 = \dim R - 1.$$

So it is enough to show $\dim R/xR < \dim R$. Suppose not. Then there exists a minimal prime P of R such that $x \in P$ and $\dim R/P = \dim R$. As P is minimal over $(0) = \operatorname{Ann}_R M$, $P \in \operatorname{Ass}_R M$. Hence $P = (0 :_R f)$ for some homogeneous element $f \in M$. As $x \in P$, $f \in (0 :_M x)$ and so certainly $(R_+)^k f \in (0 :_M x)$ for all $k \geq 1$. Since $(0 :_M x)_n = 0$ for large n, we see that $(R_+)^k f = 0$ for some k. Hence $R_+ \subseteq P$ and so

$$\dim M = \dim R = \dim R/P \leq \dim R/R_+ = \dim R_0,$$

a contradiction. Thus, $\dim R/xR < \dim R$.

Theorem 7.2. *Let R be a nonnegatively graded Noetherian ring with R_0 Artinian local and M a non-zero finitely generated graded R-module of dimension d. Then there exists positive integers $s_1, \ldots, s_d$ and $g(t) \in \mathbf{Z}[t, t^{-1}]$ where $g(1) \neq 0$ such that*

$$P_M(t) = \frac{g(t)}{\prod_{i=1}^{d}(1 - t^{s_i})}$$

proof. We again use induction on $d = \dim M$. If $d = 0$ then $M_n = 0$ for all but finitely many n and thus $g(t) = P_M(t) \in Z[t, t^{-1}]$. Note $g(1) = \lambda(M) \neq 0$.

Suppose $d > 0$. Let $x \in R_+$ be a superficial element for M and let $s_d = \deg x$. Consider the exact sequence

$$0 \longrightarrow (0 :_M x)_n \longrightarrow M_n \xrightarrow{x} M_{n+s_d} \longrightarrow (M/xM)_{n+s_d} \longrightarrow 0.$$

Since length is additive on exact sequences, we get that

$$(*) \qquad \lambda(M_{n+s_d}) - \lambda(M_n) = \lambda((M/xM)_{n+s_d}) - \lambda((0 :_M x)_n)$$

and so

$$\lambda(M_{n+s_d})t^{n+s_d} - \lambda(M_n)t^{n+s_d} = \lambda((M/xM)_{n+s_d})t^{n+s_d} - \lambda((0 :_M x)_n)t^{n+s_d}.$$

Summing these equations up over all n, we get that

$$P_M(t) - t^{s_d}P_M(t) = P_{M/xM}(t) - t^{s_d}P_{(0:_M x)}(t).$$

By Lemma 7.2, $\dim M/xM = d - 1$ and since $(0 :_M x)$ has finite length, $\dim(0 :_M x) = 0$ or $(0 :_M x) = 0$. Therefore, there exists $g_1(t), g_2(t) \in \mathbf{Z}[t, t^{-1}]$ with $g_1(1) \neq 0$ and positive integers $s_1, \ldots, s_{d-1}$ such that

$$(1 - t^{s_d})P_M(t) = \frac{g_1(t)}{\prod_{i=1}^{d-1}(1 - t^{s_i})} + t^{s_d}g_2(t).$$

Thus,

$$P_M(t) = \frac{g_1(t) + t^{s_d}\prod_{i=1}^{d-1}(1 - t^{s_i})g_2(t)}{\prod_{i=1}^{d}(1 - t^{s_i})}.$$

Let $g(t) = g_1(t) + t^{s_d}\prod_{i=1}^{d-1}(1 - t^{s_i})g_2(t)$. We need to show that $g(1) \neq 0$. If $d > 1$ then $g(1) = g_1(1) \neq 0$. If $d = 1$ then $g(1) = g_1(1) - g_2(1) = \lambda(M/xM) - \lambda((0 :_M x))$. Suppose $\lambda(M/xM) = \lambda((0 :_M x))$. Using equation (*) above, we see that for n sufficiently large

$$\begin{aligned}
\lambda(M_{n+1}) + \cdots + \lambda(M_{n+s_d}) &= \sum_{i=-\infty}^{n} \lambda(M_{i+s_d}) - \lambda(M_i) \\
&= \sum_{i=-\infty}^{n} \lambda(M/xM)_{i+s_d} - \sum_{i=-\infty}^{n} \lambda((0 :_M x)_n) \\
&= \lambda(M/xM) - \lambda((0 :_M x)) \\
&= 0.
\end{aligned}$$

Hence, $\lambda(M_n) = 0$ for n sufficiently large and thus $\dim M = 0$, a contradiction.

Remark 7.1. The above proof also shows that if M is nonnegatively graded then $g(t) \in \mathbf{Z}[t]$.

Corollary 7.3. *Let $R = R_0[x_1, \ldots, x_k]$ be a Noetherian graded ring with R_0 Artinian local and $\deg x_i = 1$ for all i. Let M be a non-zero finitely generated graded R-module. Then $s(M) = \dim M$. In particular, $\dim M \leq k$ and $\deg Q_M(x) = \dim M - 1$.*

proof. By Corollary 7.1 and Theorem 7.2,

$$P_M(t) = \frac{g(t)}{\prod_{i=1}^{d}(1 - t^{s_i})} = \frac{f(t)}{(1-t)^s}.$$

where $f(1) \cdot g(1) \neq 0$, $d = \dim M$ and $s = s(M)$. From this equation it is clear that $s = d$. The last statement follows from Corollaries 7.1 and 7.2.

Suppose M is a graded R-module possessing a Hilbert polynomial $Q_M(x)$. Since $Q_M(n) \in \mathbf{Z}$ for sufficiently large n, it follows that $Q_M(\mathbf{Z}) \subseteq \mathbf{Z}$. It can be shown that there exist unique integers $e_i = e_i(M)$ for $i = 0, \ldots, d-1$ such that

$$Q_M(x) = e_0\binom{x+d-1}{d-1} - e_1\binom{x+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}.$$

The integers $e_0, \ldots, e_{d-1}$ are called the *Hilbert coefficients of M.* The first coefficient, e_0, is called the *multiplicity* of M and is denoted $e(M)$. In the case $\dim M = 0$, $e(M)$ is defined to be $\lambda(M)$. (See the first exercise below.)

In Exercises 7.9–7.13, R is a Noetherian graded ring such that $R = R_0[R_1]$ and R_0 is Artinian local.

Exercise 7.9. Let M be a finitely generated graded R-module of dimension d. Show that there exists a unique integer $e_d(M)$ such that

$$\sum_{i=-\infty}^{n} \lambda(M_i) = \sum_{i=0}^{d}(-1)^i e_i(M)\binom{n+d-i}{d-i}$$

for all sufficiently large integers n. Moreover, if $d = 0$ then $e_0(M) = \lambda(M)$.

Exercise 7.10. Let M be a f.g. graded R-module of positive dimension and $x \in R_+$ a superficial element for M. If $\dim M > 1$ prove that

$$e(M/xM) = \deg x \cdot e(M).$$

If $\dim M = 1$ prove that

$$e(M/xM) = \lambda(M/xM) = \deg x \cdot e(M) - \lambda((0 :_M x)).$$

Exercise 7.11. Let M be a non-zero f.g. graded R-module. Prove that $e(M) > 0$.

Exercise 7.12. Let M be a f.g. graded R-module of dimension $d \geq 2$ and $x \in R_1$ a superficial element for M. Prove that $e_i(M/xM) = e_i(M)$ for $i = 1, \ldots, d-2$.

Definition 7.3. Let (R, m) be a local ring and I an m-primary ideal. Applying Exercise 7.9 to the graded ring $gr_I(R)$, we see that that the function $h_I(n) = \lambda(R/I^n)$ (called the *Hilbert function* of I) coincides for large n with a polynomial $q_I(n) \in \mathbf{Q}[n]$ (the *Hilbert polynomial* of I). We often write $q_I(n)$ in the following form:

$$q_I(n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n+d-i-1}{d-i},$$

where $d = \dim gr_I(R)$ (but see Theorem 7.3 below). The integers $e_0(I), \ldots, e_d(I)$ are called the *Hilbert coefficients* of I.

Lemma 7.3. *Let (R, m) be a local ring and I and J two m-primary ideals of R. Then $\dim gr_I(R) = \dim gr_J(R)$.*

proof. Without loss of generality, we may assume $J = m$. As I is m-primary there exists an integer k such that $m^k \subseteq I$. Then for all n, $m^{kn} \subseteq I^n \subseteq m^n$. Thus, for n sufficiently large, $q_m(kn) \leq q_I(n) \leq q_m(n)$. Hence, $\dim gr_I(R) = \deg q_I(n) = \deg q_m(n) = \dim gr_m(R)$.

Theorem 7.3. *Let (R, m) be a local ring and I an m-primary ideal. Then $\dim gr_I(R) = \dim R$.*

proof. By Lemma 7.3, it is enough to prove the result in the case I is generated by a system of parameters $x_1, \ldots, x_d$, where $d = \dim R$. Since $gr_I(R) = R/I[x_1^*, \ldots, x_d^*]$, where x_i^* is the image of x_i in I/I^2, we know by Corollary 7.3 that $\dim gr_I(R) \leq d$. Let $e = \dim gr_I(R)$ and M the homogeneous maximal ideal of $gr_I(R)$. By Corollary 5.4, there exists homogeneous elements $w_1, \ldots, w_e \in gr_I(R)$ of positive degree such that $M = \sqrt{(w_1, \ldots, w_e)}$. Furthermore, by replacing the w_i's with appropriate powers of them, we may assume that each w_i has the same degree p. For each i, let $u_i \in I^p$ be such that $u_i^* = w_i$. Then for n sufficiently large, $I^n/I^{n+1} = [(u_1, \ldots, u_e)I^{n-p} + I^{n+1}]/I^{n+1}$. Thus, $I^n = (u_1, \ldots, u_e)I^{n-p} \subseteq (u_1, \ldots, u_e)$. Hence, $u_1, \ldots, u_e$ is an s.o.p. for R and so $e \geq d$.

Exercise 7.13. Let (R, m) be a Cohen-Macaulay local ring of dimension d and J an ideal generated by a system of paramaters for R. Prove that for all $n \geq 1$,

$$h_J(n) = q_J(n) = \lambda(R/J)\binom{n+d-1}{d}.$$

Hence, $e_0(J) = \lambda(R/J)$ and $e_i(J) = 0$ for $i \geq 1$. (Hint: first find the Hilbert polynomial of $gr_J(R)$.)

Exercise 7.14. Let R be a graded ring and

$$0 \to M_k \to M_{k-1} \to \cdots \to M_0 \to 0$$

an exact sequence of graded R-modules with degree 0 maps. Suppose each M_i has a Hilbert polynomial. Prove that $\sum_i (-1)^i Q_{M_i}(x) = 0$.

Proposition 7.4. *Let R be a Noetherian graded ring and M a non-zero finitely generated graded R-module. Then there exists a filtration*

$$(*) \qquad 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that for $1 \le i \le r$, $M_i/M_{i-1} \cong (R/P_i)(\ell_i)$ for some homogeneous prime P_i and integer ℓ_i. We'll refer to such a filtration as a quasi-composition series *for M. The set $S = \{P_1, \dots, P_r\}$ is not uniquely determined by M, but $\operatorname{Min}(S) = \operatorname{Min}_R M$. If $p \in \operatorname{Min}_R M$, then the number of times p occurs in any quasi-composition series for M is $\lambda_{R_p}(M_p)$.*

proof. We first show the existence of a quasi-composition series for M. Let

$$\Lambda = \{N \mid N \text{ a graded submodule of } M \text{ which is zero or has a quasi-composition series}\}.$$

Let N be a maximal element of Λ. Suppose $M \neq N$ and let $N' = M/N$. Choose $q \in \operatorname{Ass}_R N'$. Then N' has a graded submodule L isomorphic to $(R/q)(\ell)$ for some integer ℓ. Let M' be the inverse image of L in M. Then M' contains N properly and $M' \in \Lambda$, a contradiction.

Now, $p \supseteq \operatorname{Ann}_R M$ if and only if $p \supseteq \operatorname{Ann}_R M_i/M_{i-1}$ for some i, which holds if and only if $p \supseteq P_i$ for some i. Thus, $\operatorname{Min}(S) = \operatorname{Min}_R M$.

Let $p \in \operatorname{Min}_R M$. Then $(M_i/M_{i-1})_p = 0$ or R_p/pR_p. Hence, localizing (*) at p yields a composition series for the R_p-module M_p, and $\lambda_{R_p}(M_p)$ is the number of nonzero factors, i.e., the number of times $(R/p)(\ell)$ occurs as a factor in (*).

Theorem 7.4. *(The Associativity Formula) Let $R = R_0[R_1]$ be a Noetherian graded ring such that R_0 is an Artinian local ring. Let M be a non-zero finitely generated graded R-module of dimension n. Then*

$$e(M) = \sum_{\dim R/p = n} \lambda(M_p) \cdot e(R/p).$$

proof. If $n = 0$ then $e(M) = \lambda(M)$ and the formula holds (since $\lambda_{R_m}(M_m) = \lambda_R(M)$ and $e(R/m) = \lambda(R/m) = 1$, where m is the homogeneous maximal ideal of R). So we may assume $n > 0$. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ be a quasi-composition series for M, where $M_i/M_{i-1} \cong (R/P_i)(\ell_i)$ for $1 \le i \le r$. Using the exact sequences $0 \to M_{i-1} \to M_i \to (R/P_i)(\ell_i)$ and Exercise 7.14, we obtain

$$Q_M(x) = \sum_{i=1}^{r} Q_{R/P_i}(x + \ell_i).$$

Since $e(M)$ is the coefficient of x^{n-1} in $(n-1)! \cdot Q_M(x)$ and $\dim R/P_i \leq n$, we see that

$$e(M) = \sum_{\dim R/P_i = n} e(R/P_i).$$

(Note that the leading coefficients of $Q_{R/P_i}(x)$ and $Q_{R/P_i}(x+\ell_i)$ are equal.) By Proposition 7.4, if $\dim R/P_i = n$ then $P_i \in \operatorname{Min}_R M$ and conversely, if $\lambda(M_p) \neq 0$ and $\dim R/p = n$ then $p = P_i$ for some i. Finally, if $\dim R/p = n$ then the number of times R/p occurs as a factor (up to a shift) in any quasi-composition series for M is $\lambda_{R_p}(M_p)$.

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§8. Reductions and integral closures of ideals

Definition 8.1. Let R be a Noetherian ring and I an ideal of R. An ideal $J \subseteq I$ is called a *reduction* of I if $JI^r = I^{r+1}$ for some $r \geq 0$. (The case when $r = 0$ just means $J = I$.) The smallest such r is called the *reduction number* of I with respect to J and is denoted $r_J(I)$.

Example 8.1. Let $R = k[x, y]$ and $I = (x^3, x^2y, y^3)$. Then $J = (x^3, y^3)$ is a reduction of I and $r_J(I) = 2$, since $I^3 = JI^2$ and $x^4y^2 \in I^2 \setminus JI$.

Example 8.2. Let $R = k[t^3, t^4, t^5]$ and $I = (t^3, t^4, t^5)$. Then $J = (t^3)$ is a reduction of I and $r_J(I) = 1$.

Exercise 8.1. Let $R = k[x, y]$ and $I = (x^4, x^3y, x^2y^2, y^4)$. Find a reduction of I which has two generators.

Exercise 8.2. Suppose J is a reduction of I. Prove that $\sqrt{J} = \sqrt{I}$.

Exercise 8.3. Suppose (R, m) is a local ring and I an m-primary ideal. Let J be a reduction of I. Prove that $e_0(I) = e_0(J)$.

Definition 8.2. An element $a \in R$ is said to be *integral* over an ideal I if there exists an equation of the form

$$a^n + r_1a^{n-1} + \cdots + r_{n-1}a + r_n = 0,$$

where $r_i \in I^i$ for all i.

Example 8.3. Let $R = k[x, y, z]$ and $I = (x^3, y^3, z^3)$. Then xyz is integral over I, since $(xyz)^3 - x^3y^3z^3 = 0$ and $x^3y^3z^3 \in I^3$.

Exercise 8.4. Let $R = k[x, y, z]/(x^4 - y^2z^2)$ and $I = (y^2, z^2)$. Show that xy is integral over I.

Exercise 8.5. Prove that a is integral over I if and only if $at \in R[t]$ is integral over the subring $R[It]$.

Lemma 8.1. *Let I be an ideal of R. Then the set of all elements of R integral over I forms an ideal $\overline{I}$, called the* integral closure *of I.*

proof. Let $a, b \in \overline{I}$, and $r \in R$. We need to show $a + b \in \overline{I}$ and $ra \in \overline{I}$. By the previous exercise, it is enough to show that $(a + b)t$ and rat are integral over the ring $R[It]$. But the integral closure of $R[It]$ in $R[t]$ forms a subring of $R[t]$. Therefore, we are done since at and bt are integral over $R[It]$.

Exercise 8.6. Let R be a graded ring and I a homogeneous ideal. Prove that $\overline{I}$ is homogeneous. (Hint: consider the ring $R[It]$ as bigraded and apply Theorem 6.1.)

Exercise 8.7. *Prove that $\overline{\overline{I}} = \overline{I}$ and that $\overline{I} \cdot \overline{J} \subseteq \overline{IJ}$.*

Theorem 8.1. *Let R be a ring, $J \subseteq I$ ideals of R. The following are equivalent:*

1. *J is a reduction of I.*
2. *I is integral over J; i.e., $I \subseteq \overline{J}$.*

proof. Let $T = R[It]$ and $S = R[Jt]$. Then I is integral over J if and only if T is integral over S. And J is a reduction of I if and only if for some $n \geq 0$, $J(T_+)^n = (T_+)^{n+1}$. The result now follows from Theorem 6.2.

Definition 8.3. Let (R, m) be a local ring. The elements $a_1, \ldots, a_r$ are said to be *analytically independent* if whenever $\phi(T_1, \ldots, T_r) \in R[T_1, \ldots, T_r]$ is a homogeneous polynomial such that $\phi(a_1, \ldots, a_r) = 0$, then all the coefficients of ϕ are in m; i.e., $\phi \in m[T_1, \ldots, T_n]$

Exercise 8.8. Let (R, m) be a local ring and $I = (a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are analytically independent. Prove that $\mu(I) = n$.

Theorem 8.2. Let $I = (a_1, \ldots, a_r)$. Then the following are equivalent:

1. $a_1, \ldots, a_r$ are analytically independent.
2. $R[It]/mR[It]$ is isomorphic to a polynomial ring over a field in r variables.
3. $\dim R[It]/mR[It] = r$.

proof. (1) $\iff$ (2): define $g \colon R[T_1, \ldots, T_r] \to R[It]/mR[It]$ by $g(f(\underline{T})) = \overline{f(a_1t, \ldots, a_rt)}$. Clearly, g is surjective and $mR[T_1, \ldots, T_r] \subseteq \ker g$. It is enough to show that $a_1, \ldots, a_r$ are analytically independent if and only if $\ker g = mR[T_1, \ldots, T_r]$. The "if" direction is true by definition. For the converse, suppose $a_1, \ldots, a_r$ are analytically independent. Let f be a homogeneous element in $\ker g$. Then $f(a_1t, \ldots, a_rt) \in mR[It]$. Now, it is easily seen that $mR[It] = \{h(a_1t, \ldots, a_rt) \mid h \in mR[T_1, \ldots, T_r]\}$. Hence, $f(a_1t, \ldots, a_rt) = h(a_1t, \ldots, a_rt)$ for some homogeneous $h \in mR[T_1, \ldots, T_r]$. Now $0 = (f - h)(a_1t, \ldots, a_rt)) = t^d(f - h)(a_1, \ldots, a_r)$, where $d = \deg f = \deg h$. Hence, $(f - h)(a_1, \ldots, a_r) = 0$ and so, by definition of analytic independence, $f - h \in mR[T_1, \ldots T_r]$. Thus, $f \in mR[T_1, \ldots, T_r]$.

(2) $\iff$ (3): Consider the map g defined above. Since $mR[T_1, \ldots, T_r]$ is contained in $\ker g$, we see that $R[It]/mR[It]$ is the homomorphic image of $R/m[T_1, \ldots, T_r]$. Thus, $\dim R[It]/mR[It] \leq r$ with equality if and only if $R/m[T_1, \ldots, T_r] \cong R[It]/mR[It]$.

Corollary 8.1. *Let R be a ring and $x_1, \dots, x_r$ a regular sequence. Then $x_1, \dots, x_r$ are analytically independent.*

proof. Let $I = (x_1, \dots, x_r)$. Then $R[It]/IR[It] = gr_I(R) \cong (R/I)[T_1, \dots, T_r]$, where $T_1, \dots, T_r$ are indeterminates corresponding to the images of $x_1, \dots, x_r$ in the first graded piece of $gr_I(R)$. (This is a fact about regular sequences. See Theorem 16.2 of [Mats], for example.) Therefore, $R[It]/mR[It] \cong R/m[T_1, \dots, T_r]$. Thus, $x_1, \dots, x_r$ are analytically independent.

Corollary 8.2. *Let (R, m) be a local ring and $x_1, \dots, x_d$ a system of parameters for R. Then $x_1, \dots, x_d$ are analytically independent.*

proof. Let $I = (x_1, \dots, x_d)$. By Theorem 7.3, $\dim R[It]/IR[It] = \dim gr_I(R) = d$. As $m^n \subseteq I$ for some n, $(mR[It])^n \subseteq IR[It]$. Thus, $\dim R[It]/mR[It] = \dim R[It]/IR[It] = d$, so $x_1, \dots, x_d$ are analytically independent.

Definition 8.4. Let $J \subseteq I$ be ideals where J is a reduction of I. Then J is called a *minimal reduction* of I if no reduction of I is properly contained in J. An ideal I is called *basic* if it is a minimal reduction of itself.

Exercise 8.9. Let $J \subseteq I$ be ideals where J is a reduction of I. Prove that J is a minimal reduction of I if and only if J is basic.

Exercise 8.10. Let (R, m) be a local ring and $J \subseteq I$ ideals of R. Then J is a reduction of I if and only if $J + mI$ is a reduction of I.

Proposition 8.1. *(Northcott-Rees) Let (R, m) be a local ring and I an ideal of R. Then I has a minimal reduction.*

proof. Let $P = \{(J + mI)/mI \mid J \text{ is a reduction of } I\}$. Since P consists of subspaces of a finite dimensional vector space, P has a minimal element $(K + mI)/mI$. Let $a_1, \dots, a_s$ be elements of K such that the images of these elements in $(K + mI)/mI$ form a vector space basis. Let $J = (a_1, \dots, a_s)$. Then $J + mI = K + mI$. By Exercise 6.10, J is a reduction of I. We claim that J is a minimal reduction of I. First note that $J \cap mI = mJ$. Clearly $mJ \subseteq J \cap mI$. Let $c \in J \cap mI$. Then $c = r_1a_1 + \cdots r_sa_s \in mI$ for some $r_1, \dots, r_s \in R$. As the images of $a_1, \dots, a_s$ in I/mI are linearly independent over R/m, we see that each $r_i \in m$. Hence, $J \cap mI \subseteq mJ$. Now suppose $L \subseteq J$ is a reduction of I. We need to show that $L = J$. Since $L + mI \subseteq J + mI$ and $J + mI$ is a minimal element of P, we must have $L + mI = J + mI$. In particular, $J \subseteq L + mI$. Now let $u \in J$. Then $u = x + y$ for some $x \in L$ and $y \in mI$. Hence $y = u - x \in J \cap mI = mJ$. Thus, $J \subseteq L + mJ$ and so $L = J$ by Nakayama.

Exercise 8.11. Let (R, m) be a local ring and J a minimal reduction of I. Prove that every minimal generating set for J can be extended to a minimal generating set for I. (Hint: use the method of proof of Proposition 8.1.)

Definition 8.5. Let (R, m) be a local ring and I an ideal of R. The *analytic spread* of I, denoted $\ell(I)$, is defined to be $\dim R[It]/mR[It]$.

Exercise 8.12. Let (R, m) be a local ring and I an m-primary ideal. Prove that $\ell(I) = \dim R$.

Proposition 8.2. *Let (R, m) be a local ring and $J \subseteq I$ ideals such that I is integral over J. Then $\ell(I) = \ell(J)$.*

proof. Let $S = R[It]$ is integral over $T = R[Jt]$, $\sqrt{mS \cap T} = \sqrt{mT}$. Hence $\ell(I) = \dim S/mS = \dim T/(mS \cap T) = \dim T/mT = \ell(J)$.

Theorem 8.3. Let (R, m) be a local ring such that R/m is infinite. Let $J \subseteq I$ be a reduction of I. Then the following are equivalent:

(1) J is a minimal reduction of I.
(2) J is generated by analytically independent elements.
(3) $\mu(J) = \ell(I)$.

proof. (1)⇒ (2): Consider the ring $S = R[Jt]/mR[Jt]$. By Corollary 6.1, there exists $w_1, \ldots, w_r \in J \setminus mJ$ such that S is integral over $T = (R/m)[\overline{w_1 t}, \ldots, \overline{w_r t}]$ (where $\overline{w_i t}$ is the image of $w_i t$ in Jt/mJt), and T is isomorphic to a polynomial ring in r variables over R/m. By Theorem 6.2, $(S_+)^{n+1} = (\overline{w_1 t}, \ldots, \overline{w_r t})(S_+)^n$ for some $n \geq 0$. That is, $J^{n+1}/mJ^{n+1} = ((w_1, \ldots, w_r)J^n + mJ^{n+1})/mJ^{n+1}$ for some $n \geq 0$. By Nakayama's lemma, this means that $J^{n+1} = (w_1, \ldots, w_r)J^n$, and so $(w_1, \ldots, w_r)$ is a reduction of J. As J is a minimal reduction of I, J is basic by Exercise 8.9. Thus, $J = (w_1, \ldots, w_r)$ and $S = T$. Hence J is generated by analytically independent elements by Theorem 8.2.

(2)⇒ (1): By Exercise 8.9, it is enough to show that J is basic. Let $K \subseteq J$ be a minimal reduction of J. By (1)⇒(2), K is generated by analytically independent elements. Furthermore, $\ell(I) = \ell(J)$, so $S = R[Jt]/mR[Jt]$ and $T = R[Kt]/mR[Kt]$ are isomorphic to polynomial rings over R/m of the same dimension. Thus, $S \cong T$ as graded rings. Now, since $T = R[Jt]/mR[Jt] = R[Kt]/(mR[Jt] \cap R[Kt])$ (see the proof of Proposition 8.2). Hence, $\lambda(J/mJ) = \lambda(S_1) = \lambda(T_1) = \lambda(K/(mJ \cap K)) = \lambda((K + mJ)/mJ)$. Thus, $J = K + mJ$ and so $J = K$ by Nakayama.

(2)⇒ (3): By Theorem 8.2, Exercise 8.8, and Prop. 8.1, $\mu(J) = \dim R[Jt]/mR[Jt] = \ell(J) = \ell(I)$.

(3)⇒ (2): By Proposition 8.1, $\mu(J) = \ell(J) = \dim R[Jt]/mR[Jt]$. Hence $R[Jt]/mR[Jt]$ is isomorphic to a polynomial ring in $\mu(J)$ variables over R/m. Thus, J is generated by analytically independent elements by Theorem 8.2.

Exercise 8.13. Let (R, m) be a local ring with infinite residue field and let I be an ideal of R. Prove that $\operatorname{ht} I \leq \ell(I) \leq \mu(I)$.

§9. Graded free resolutions

Throughout this section R will denote a nonnegatively graded Noetherian ring such that R_0 is local. We let m denote the homogeneous maximal ideal of R.

Proposition 9.1. *(Graded version of Nakayama's Lemma) Let M be a finitely generated graded R-module. Then $\mu_R(M) = \dim_{R/m}(M/mM)$. Morever, there exists $\mu_R(M)$ homogeneous elements which generate M.*

proof. Choose homogeneous elements $x_1, \dots x_k \in M$ such that their images in M/mM form an R/m-basis. It suffices to show that $x_1, \dots x_k$ generate M. Let N be the submodule generated by $x_1, \dots, x_k$. Then $M = N + mM$ and hence $M/N = m(M/N)$. If $M \neq N$, then M/N is a finitely generated non-zero graded R-module. Let s be the smallest integer such that $(M/N)_s \neq 0$. Then $(M/N)_s = n(M/N)_s$, where n is the maximal of R_0. This contradicts the local version of Nakayama's lemma. Hence, $M = N$.

Exercise 9.1. In the above proposition, prove that the set $\{\deg x_1, \dots, \deg x_k\}$ is uniquely determined by M.

For a f.g graded R-module M, we let $\mu(M)$ denote $\dim_{R/m} M/nM$. Clearly, $\mu(M)$ is the minimal number of generators of M.

Corollary 9.1. *Let F be a f.g. graded free R-module. Then there exist a unique set of integers $\{n_1, \dots, n_k\}$ such that $F \cong \oplus_i R(-n_i)$ (as graded modules).*

proof. Let $k = \operatorname{rk} F = \mu(F)$. Then there exists homogeneous elements $x_1, \dots, x_k$ which generate F and hence form a basis for F. Setting $n_i = \deg x_i$, we see that $F \cong \oplus R(-n_i)$. The uniqueness of the n_i's follows from the Exercise 9.1.

Exercise 9.2. Let P be a finitely generated graded projective R-module. Prove that P is free.

Definition 9.1. Let M be a finitely generated graded R-module. An exact sequence

$$\cdots \longrightarrow F_{i+1} \xrightarrow{\partial_i} F_i \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is called a *graded free resolution* of M if each F_i is free and all the maps are degree 0. The resolution is called *minimal* if $\ker(\partial_i) \subset mF_{i+1}$ for all $i \geq 0$; equivalently, rank $F_i = \mu(\ker(\partial_{i-1}))$.

Remark 9.1. By the graded version of Nakayama's lemma, it is clear that any f.g. graded R-module possesses a minimal graded free resolution.

Lemma 9.1. *Consider the following diagram of f.g. graded R-modules and degree 0 maps:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & L & \xrightarrow{\gamma} & G & \xrightarrow{\delta} & N & \longrightarrow & 0
\end{array}$$

Assume the rows are exact, F and G are free and f is a isomorphism. Suppose that $\alpha(K) \subset mF$ and $\gamma(L) \subset mG$. Then there exist degree 0 isomorphisms $g : K \to L$ and $h : F \to G$ making the diagram commute.

proof. Since F is free, it is easily seen that there exists a degree 0 maps h and g making the diagram commute. If we show that h is an isomorphism, then g must be an isomorphism by the Snake Lemma. But if we tensor the diagram with R/m, we see from the minimality that $\beta \otimes 1$ and $\delta \otimes 1$ are isomorphisms. Consequently, $h \otimes 1$ is an isomorphism. Thus, if $C = \operatorname{coker}(h)$ then $C/nC = 0$ and so $C = 0$ by (graded) Nakayama. Hence, h is surjective. Since G is free, h splits and so $\ker(h)/m\ker(h) = 0$. Therefore, $\ker(h) = 0$ and h is an isomorphism.

Theorem 9.1. *Let $F_.$ and $G_.$ be two minimal graded free resolutions of a f.g. graded R-module M. Then there exists a chain map $f_. : F_. \to G_.$ such that for each i, $f_i : F_i \mapsto G_i$ is a degree 0 isomorphism.*

proof. We define the map $f_i : F_i \to G_i$ by induction on i. To start, let $f_{-1} : M \to M$ be the identity map. Assume we have defined f_i for $i \leq p$. Let $\partial_.$ be the differential for the resolution $F_.$ and $\partial'_.$ be the differential for $G_.$. Set $K_i = \ker(\partial_{i-1})$ and $L_i = \ker(\partial'_{i-1})$ for each $i \geq 0$. We also include in our induction hypothesis that there exist isomorphisms $h_i : K_i \to L_i$ for each $i \leq p$ (this is vacuous for $p = -1$). So consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{p+1} & \longrightarrow & F_{p+1} & \longrightarrow & K_p & \longrightarrow & 0 \\
 & & & & & & \Big\downarrow{\scriptstyle h_p} & & \\
0 & \longrightarrow & L_{p+1} & \longrightarrow & G_{p+1} & \longrightarrow & L_p & \longrightarrow & 0
\end{array}$$

By the previous lemma, there exist degree 0 isomorphisms $f_{p+1} : F_{p+1} \to G_{p+1}$ and $h_{p+1} : K_{p+1} \to L_{p+1}$ which make the diagram commute.

Theorem 9.2. *Let M be a f.g. R-module and $F_.$ a minimal graded free resolution of M. For each $i \geq 0$, let*

$$F_i = \oplus_{j=1}^{r_i} R(-n_{ij}).$$

Then

(a) *For each $i \geq 0$, the set $\{n_{ij}\}_{j=1}^{r_i}$ is uniquely determined.*
(b) *For each $i, j > 0$, there exists j' such that $n_{ij} \geq n_{(i-1)j'}$ (or $n_{ij} > n_{(i-1)j'}$ if R_0 is a field).*
(c) *If $F_.$ is finite (i.e., $F_i = 0$ for i sufficiently large) and R_0 is Artinian, then*

$$P_M(t) = \sum_{i,j} (-1)^i t^{n_{ij}} P_R(t).$$

proof. By the preceeding theorem, any two minimal graded free resolutions of M are chain isomorphic with degree 0 maps, so the ranks of the F_i and the integers $\{n_{ij}\}$ are uniquely determined. This proves (a).

For $i > 0$, let $e_1, \ldots, e_r$ and $f_1, \ldots, f_s$ be homogeneous bases for F_i and F_{i-1}, respectively, where $r = r_i$ and $s = r_{i-1}$. By part (a), we can arrange the bases so that $\deg e_j = n_{ij}$ and $\deg f_j = n_{(i-1)j}$. Since $F_.$ is minimal, for each j there exist homogeneous elements $u_1, \ldots u_s \in n$ such that $\partial_{i-1}(e_j) = \sum_{j'} u_{j'} f_{j'}$. Then $n_{ij} = \deg \partial_{i-1}(e_j) = \deg u_{j'} f_{j'} \geq n_{(i-1)j'}$ for some j'. Notice that if R_0 is a field, then $\deg u_{j'} > 0$ and so $n_{ij} > n_{(i-1)j'}$.

For part (c), we have

$$\begin{aligned} P_M(t) &= \sum_i (-1)^i P_{F_i}(t) \\ &= \sum_i (-1)^i \sum_{j=1}^{r_i} P_{R(-n_{ij})}(t) \\ &= \sum_{i,j} (-1)^i t^{n_{ij}} P_R(t). \end{aligned}$$

Exercise 9.3. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field with $\deg x_i = 1$ for all i and let M be a f.g. graded R-module. Prove that M has a finite minimal graded free resolution. (Hint: use what you know about minimal resolutions over R_m where $m = (x_1, \ldots, x_d)$.)

Exercise 9.4. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field with $\deg x_i = 1$ for all i and let M be a f.g. graded R-module. Prove that the Hilbert polynomial of M is given by

$$Q_M(x) = \sum_{i,j} (-1)^i \binom{x + d - n_{ij} - 1}{d - 1}$$

where the integers n_{ij} are the invariants which occur in a minimal graded free resolution of M.